Math 103: Algebra I

ECE 111 Introduction to ECE

Week #2

Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

Objectives

- Scripts in Matlab
- Functions in Matlab
- Plotting in Matlab
- Solving $f(x) = 0$

In this lecture, we will be covering

- Rules of Algebra: valid ways to manipulate mathematical equations
- Plotting mathematical relationships,
- Solving a mathematical equation using graphical techniques, and
- Solving a mathematical equation using numerical techniques.

Algebra

Algebra I focuses on solving one equation for one unknown.

Example: Thermistor (resistor which changes with temperature)

- $R = 1000 \cdot \exp\left(\frac{3905}{T+273} - \frac{3905}{298}\right)\Omega$
- Given T, find R
- Given R, find T

Example: Photoresistor (resistor which changes with light)

- $R = 1000 \cdot (lux)^{-0.6}$
- Given lux, find R
- Given R, find lux

Graphical Solution

- Plot the function
- Find the solution from the graph

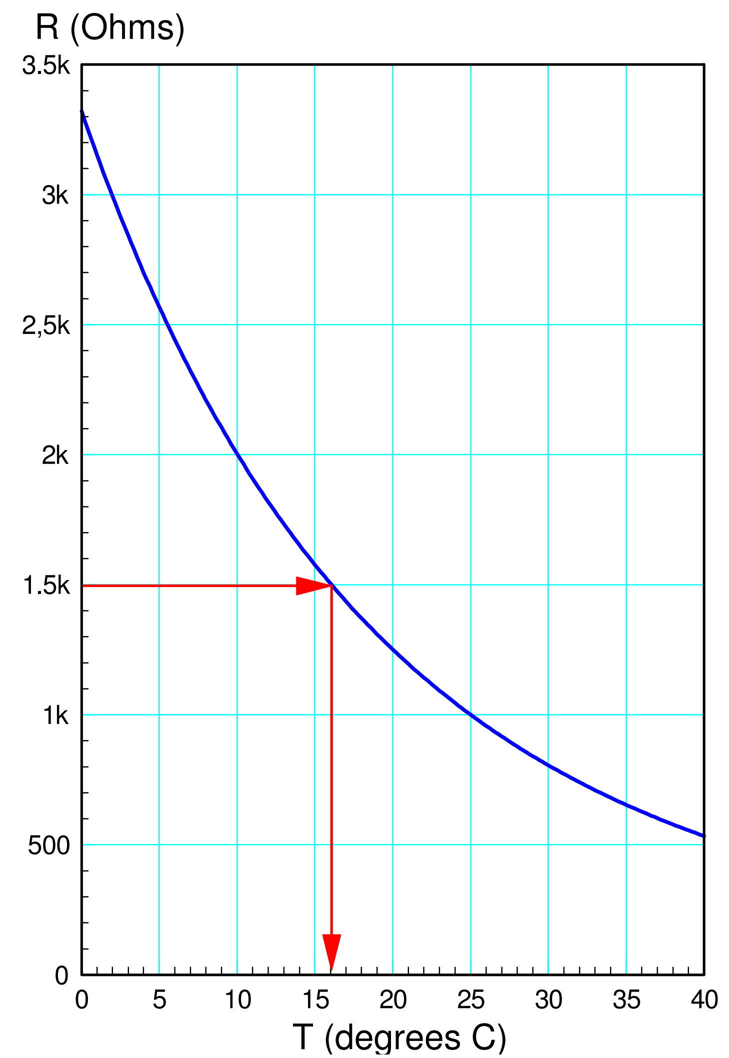
Example: Assume

$$R = 1000 \cdot \exp\left(\frac{3905}{T+293} - \frac{3905}{298}\right)\Omega$$

Find T assuming R = 1500 Ohms.

Matlab Solution: T = 16C

```
T = [0:0.01:40]';
R=1000*exp(3905./(T+273)-3905/298);
plot(T,R);
xlabel('Temperature (C)');
ylabel('Resistance (Ohms)');
grid
```



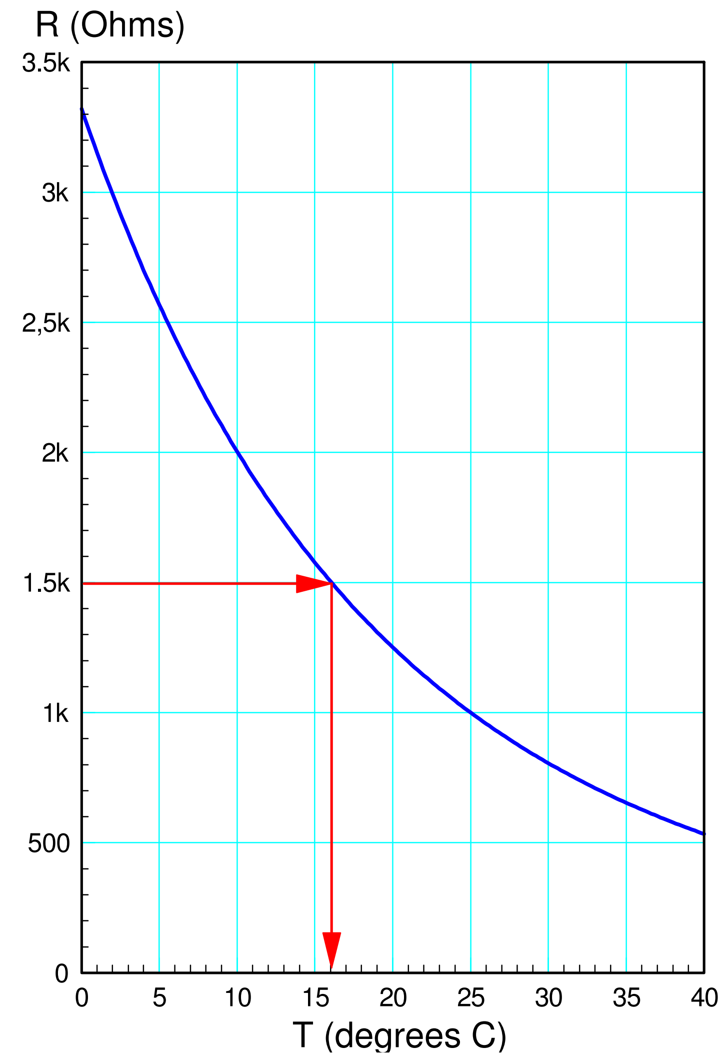
Problem: How do get more accuracy?

Option 1: Algebra

- Apply rules of algebra to determine T as a function of R

Option 2: Numerical Methods

- Iterate using Matlab



Rules of Algebra

Consider

$$A = B$$

Equals is a very powerful symbol

- It means the two sides are identical and interchangeable.
- Whatever you do on one side, do the same on the other to maintain balance

Legal Operations:

Addition:

- You can add or subtract the same value from both sides.
- Example

$$A + 5 = B + 5$$

Multiplication:

- You can multiply or divide both sides by the same number
- (except zero)

$$(A + 5) \cdot 7 = (B + 5) \cdot 7$$

Distribution:

- When multiplying stuff within parenthesis, you have to multiply each element

$$(A + 5) \cdot 7 = A \cdot 7 + 5 \cdot 7$$

Commutative Property:

- The order of addition and multiplication doesn't matter

$$A + B = B + A$$

$$A \cdot B = B \cdot A$$

Some other useful properties relate to ln() and exp()

$$\exp(x) \equiv e^x$$

$$\exp(\ln(x)) = x$$

$$\ln(\exp(x)) = x$$

Multiplying by one:

- You can multiply one side of the equation by one and still have a valid equation

$$A \cdot 1 = A$$

$$A \cdot \left(\frac{B}{B}\right) = A$$

Adding Zero: You can add zero to one side and still have a valid equation

$$A + 0 = A$$

$$A + (B - B) = A$$

Invalid Operations

Multiplying by Zero:

- This is a no-no
- Multiplying by zero makes anything work.

$$5 \cdot 0 = 3 \cdot 0$$

Dividing by zero:

- This is also a no-no:
- It also makes anything work

$$\frac{A}{0} = \frac{B}{0} = \text{undefined (or infinity)}$$

Algebra Example

Determine the value of X:

$$\left(\frac{4(x+6)-7(2x+3)}{15+2x}\right) = 25$$

Multiply both sides by (15+2x) to clear the fraction

- you can multiply both sides of an equation by the same value

$$\left(\frac{4(x+6)-7(2x+3)}{15+2x}\right)(15+2x) = 25(15+2x)$$

$$4(x+6)-7(2x+3) = 25(15+2x)$$

Multiply out each term (distributive property)

$$(4x+24)-(14x+21) = (375+50x)$$

Group terms and simplify

$$-10x + 3 = 375 + 50x$$

Add 10x to each side

$$(-10x + 3) + (10x) = (375 + 50x) + (10x)$$

$$3 = 375 + 60x$$

Subtract 375 from each side

$$3 - 375 = 375 + 60x - 375$$

$$-372 = 60x$$

Divide both sides by 60

$$\frac{-372}{60} = \frac{60x}{60} = x$$

Sidelight: Proof that 2 = 1

Using these rules, you can prove that 2 = 1. Assume

$$a = b = 1$$

Multiply both sides by a:

$$a \cdot a = ab$$

Subtract b^2 from both sides:

$$a^2 - b^2 = ab - b^2$$

note:

$$(a + b)(a - b) = a^2 + ab - ab - b^2 = a^2 - b^2$$

Rewrite the left and right sides as

$$(a + b)(a - b) = b(a - b)$$

Divide both sides by (a-b)

$$\frac{(a+b)(a-b)}{(a-b)} = \frac{b(a-b)}{(a-b)}$$

$$a + b = b \qquad \Rightarrow 1 + 1 = 2 = 1$$

Why this proof is not valid...

The problem with this proof is line 5:

$$(a+b)(a-b)=b(a-b)$$

$$2\cdot 0=1\cdot 0$$

While this is valid, canceling the zeros is not valid: you can't divide by zero

$$2\neq 1$$

Application of Algebra

Going back to the original problem, find T as a function of R

$$R = 1000 \cdot \exp\left(\frac{3905}{T+273} - \frac{3905}{298}\right)$$

Solution: Apply rules of algebra.

Divide both sides by 1000

$$\frac{R}{1000} = \exp\left(\frac{3905}{T+273} - \frac{3905}{298}\right)$$

Take the natural log of both sides

$$\ln\left(\frac{R}{1000}\right) = \frac{3905}{T+273} - \frac{3905}{298}$$

Add 3905/298 to both sides

$$\ln\left(\frac{R}{1000}\right) + \frac{3905}{298} = \frac{3905}{T+273}$$

Take the inverse of both sides

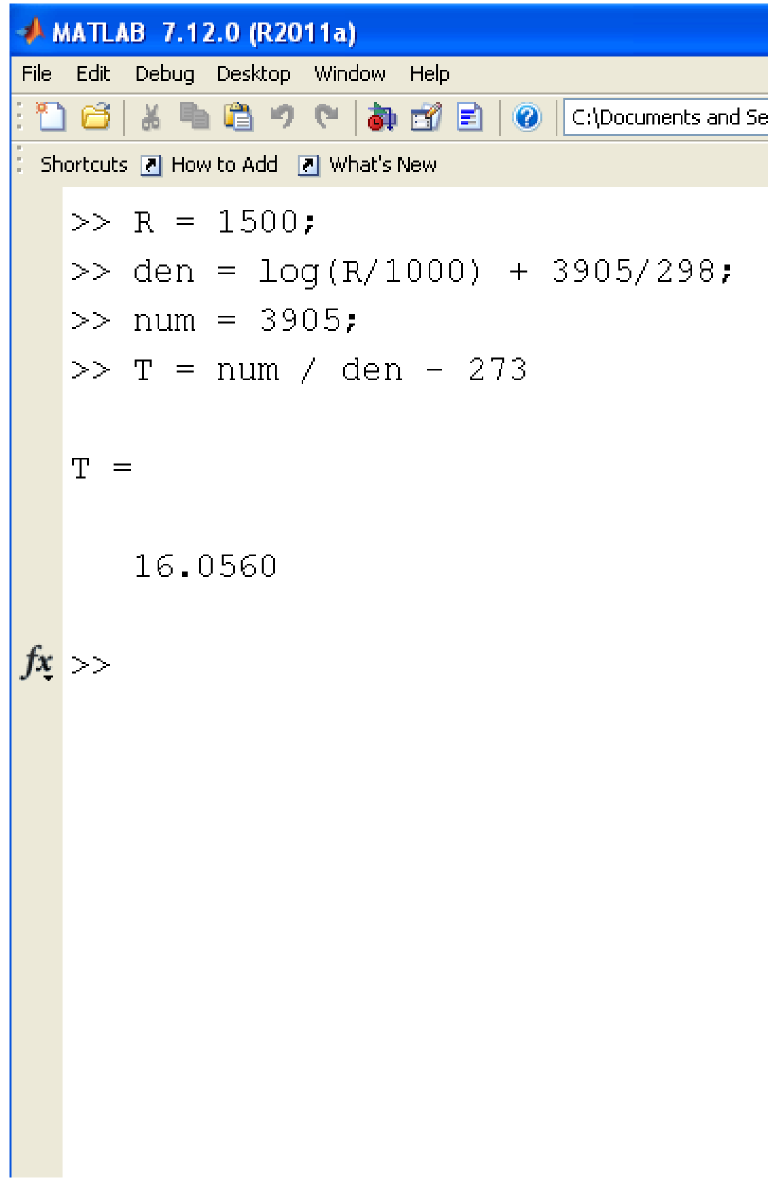
$$\left(\frac{1}{\ln\left(\frac{R}{1000}\right)+\frac{3905}{298}}\right) = \frac{T+273}{3905}$$

Multiply both sides by 3905

$$\left(\frac{3905}{\ln\left(\frac{R}{1000}\right)+\frac{3905}{298}}\right) = T+273$$

Subtract 273 from both sides

$$T = \left(\frac{3905}{\ln\left(\frac{R}{1000}\right)+\frac{3905}{298}}\right) - 273$$



Note:

- This is a lengthy process (which you'll need to do on midterms)
- Sometimes, algebra doesn't work very well...

Example: Assume (x, y) satisfy the following equations

$$y = \left(\frac{\cos(3x)}{x^2+1}\right)$$

$$y = 0.1 \cdot \exp\left(\frac{x}{2}\right)$$

Find all solutions.

Algebra doesn't work very well. Substitute for y

$$\left(\frac{\cos(3x)}{x^2+1}\right) = 0.1 \cdot \exp\left(\frac{x}{2}\right)$$

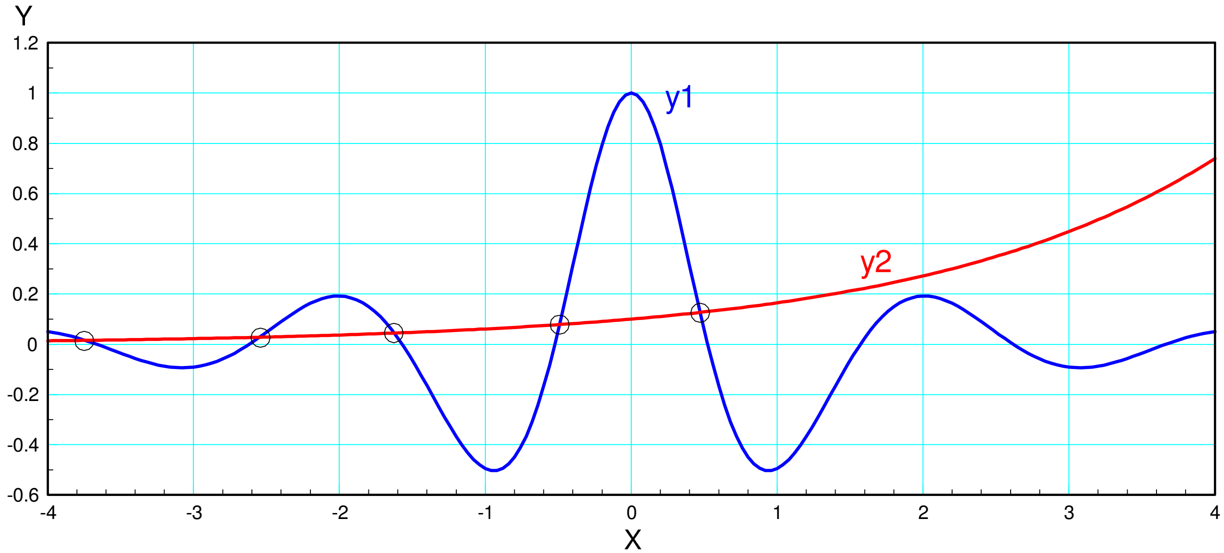
Not sure what to do now...

Graphical methods still work:

```
>> x = [-4:0.04:4]';
>> y1 = cos(3*x) ./ (x.^2 + 1);
>> y2 = 0.1*exp(x/2);
>> plot(x,y1,x,y2)
```

There are five solutions

- Graphical methods get you close
- Numeric methods to solve f(x) = 0 find these more precisely



Solving f(x) = 0 Using Numerical Techniques

- Matlab Scripts
- Matlab Functions

Scripts and functions are slightly different in Matlab:

- Scripts are similar to instructions you type in the command window. When you run a script, Matlab acts like you just typed everything in the script into the command window.
- Functions, in contrast, are subroutines you can call. For example, plot() is a function.

Unlike scripts, you cannot execute a function. Instead, it has to be called by someone else.

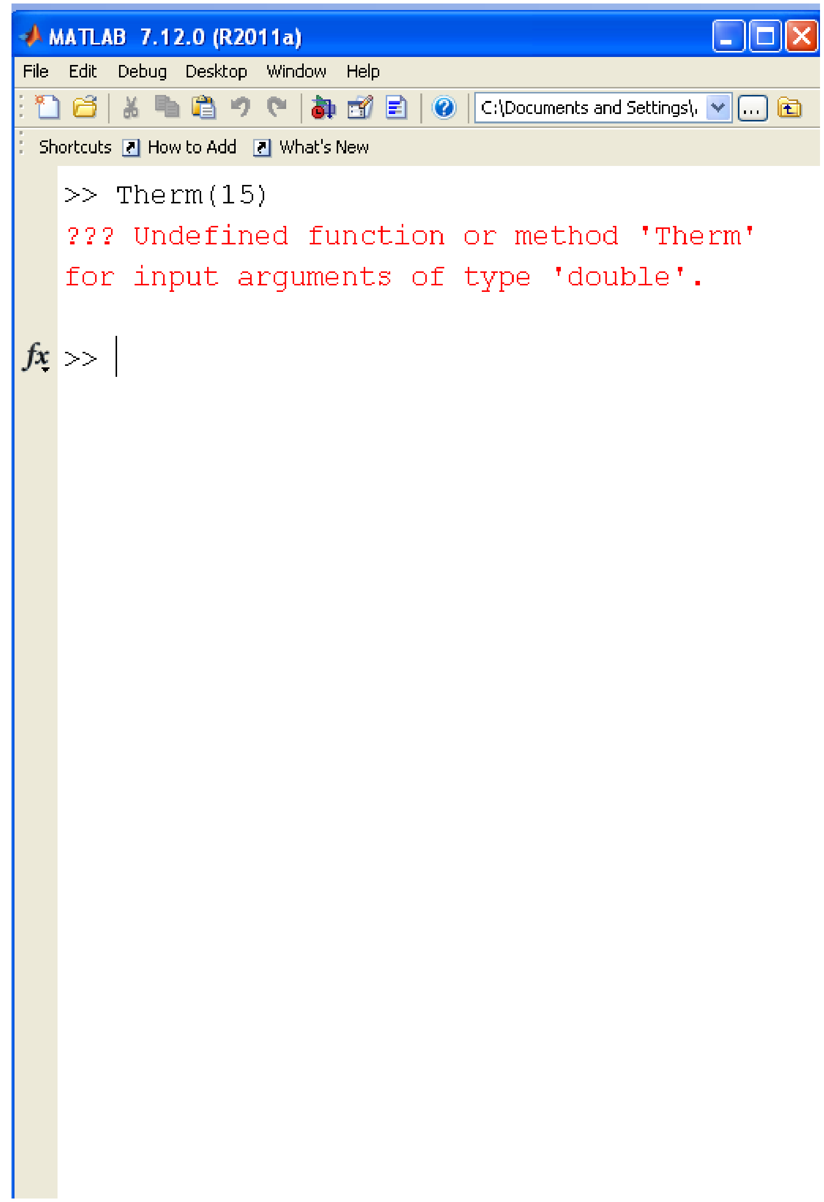
Functions in Matlab

Let's write a function called *Therm* which

- Is passed the temperature, and
- Returns the resistance of thermistor with the R-T relationship of

$$R = 1000 \cdot \exp\left(\frac{3905}{T+273} - \frac{3905}{298}\right)$$

Initially, in Matlab if you try to call this function from the command window, you'll get an error message

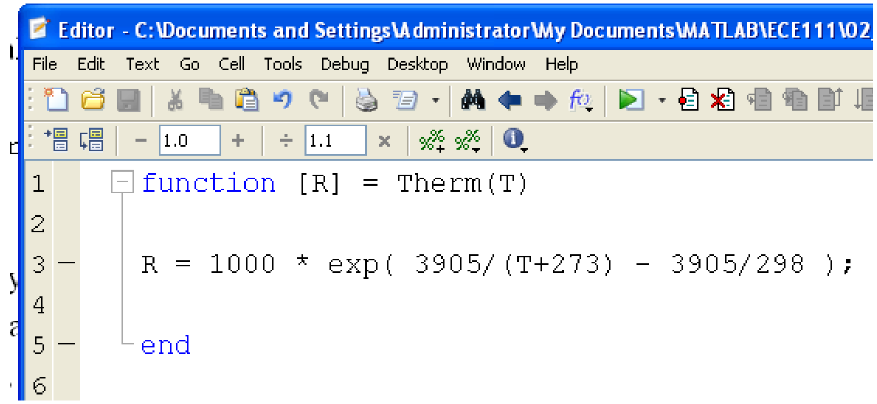


What Matlab is doing when you type in *Therm(15)* is

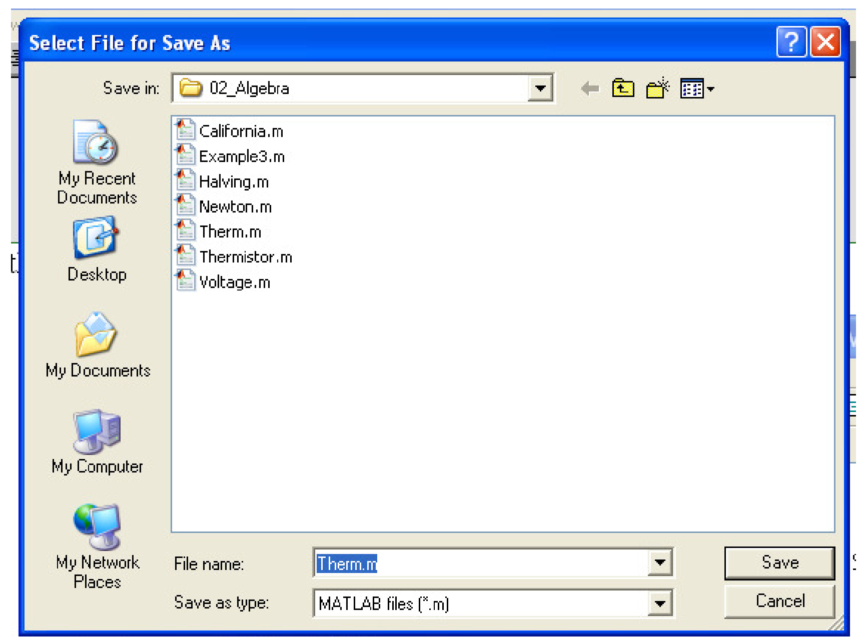
- If first checks if there is a varable called *Therm*. If so, it returns the 15th element of that array.
- If no variable *Therm* exists, it then checks if there is a file called *Therm.m* If Matlab finds that file, it then tries to call it.
- If that fails, then an error message is given: Matlab can't find *Therm* and doesn't know what to do.

Create a file *Therm.m*

- File - New Function
- Type in the following:



Now save this in the default directory with the default name, *Therm.m*



The keyword *function* tells Matlab that this is a subroutine: you cannot run it but you can call it from the command window.

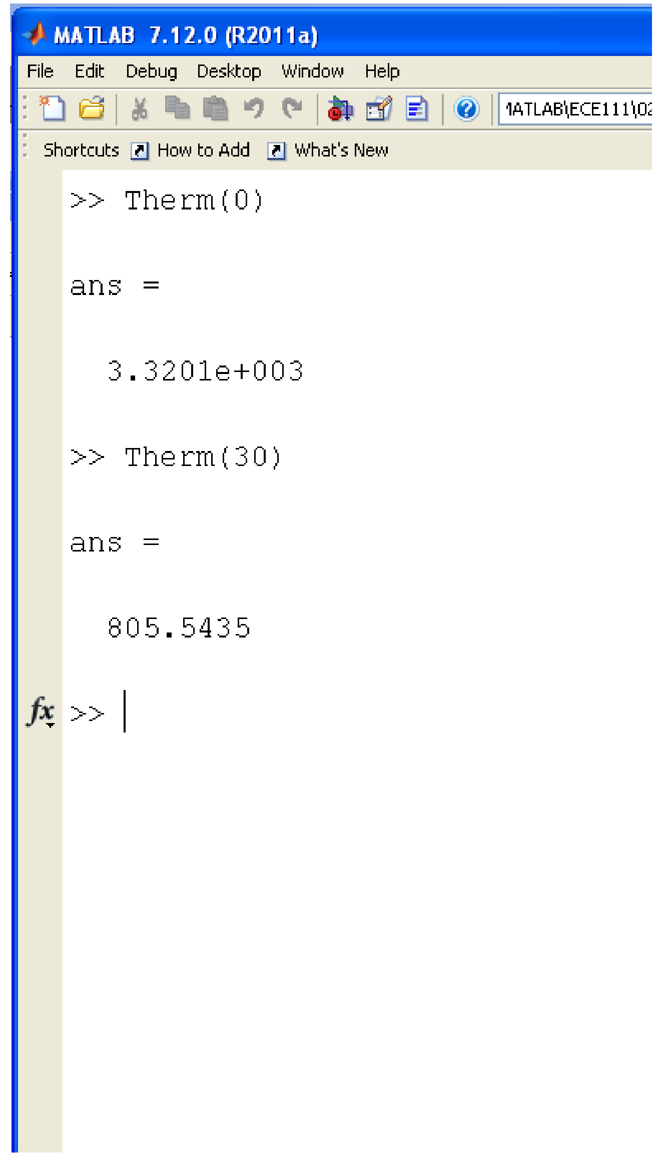
Now, you *can* call Therm.

- To find the resistance at 0C:

```
>> Therm(0)

ans =  3.3201e+003
```

- To find the resistance at 30C:

```
>> Therm(30)

ans =  805.5435
```



Solving f(x) = 0

Change the function so that the result is zero at the correct temperature

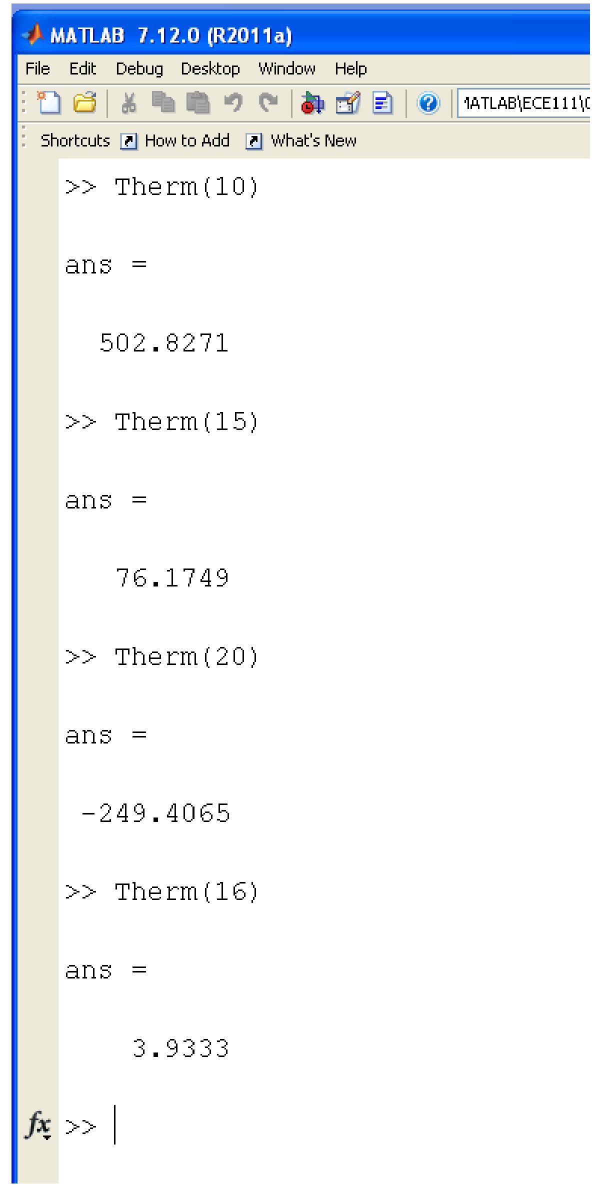
- The temperature that results in R = 1500 Ohms

```
function [e] = Therm(T)

R = 1000*exp(3905/(T+273)-3905/298);
e = R - 1500;

end
```

Guess T until e = 0

- f(x) = 0

Better methods exist for finding T:



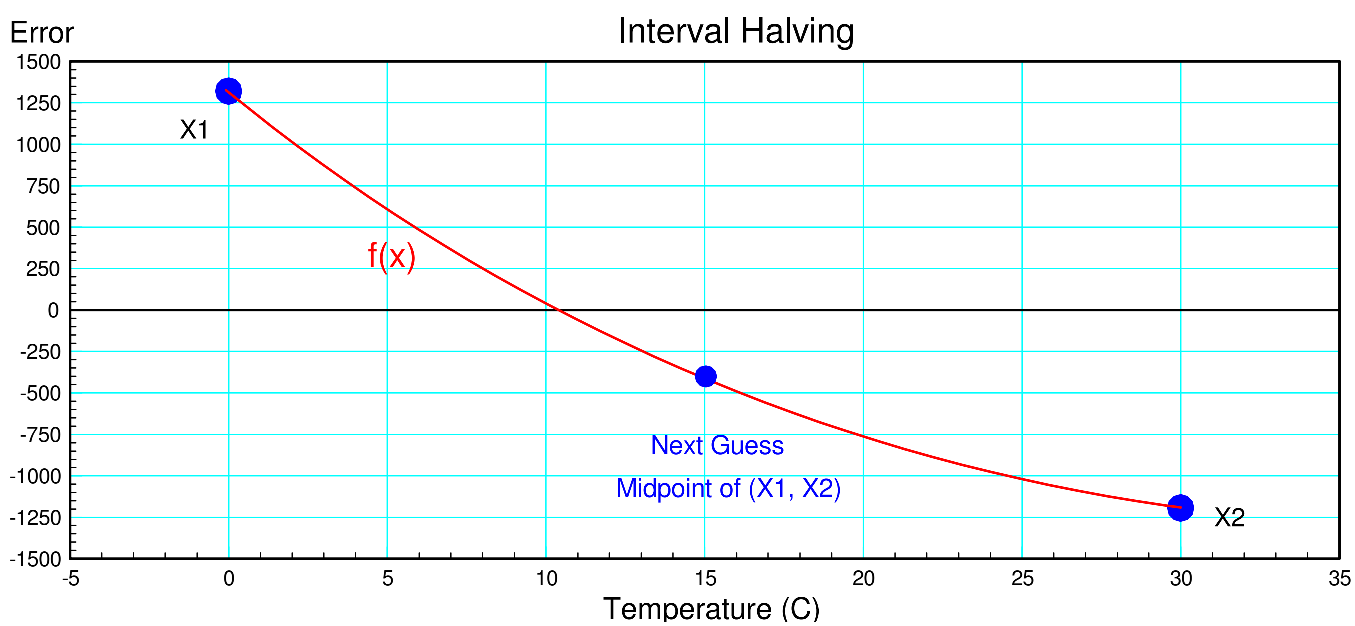
Interval Halving: Start with two guesses

- Guess #1 has a positive result (0C)
- Guess #2 has a negative result (30C)

The next guess is the midpoint between the two (+15C)

- If this result is positive, replace guess #1
- If the result is negative, replace guess #2

Repeat



Interval Halving in Action

- Iterates fifteen times
- Result: T = 16.0556

Matlab Script

```
X1 = 0;
X2 = 30;

for n=1:15
   X3 = (X1+X2)/2;
   Y3 = Therm(X3);

   if(Y3 > 0)
      X1 = X3;
   else
      X2 = X3;
   end

   disp([n X3, Y3]);
end
```

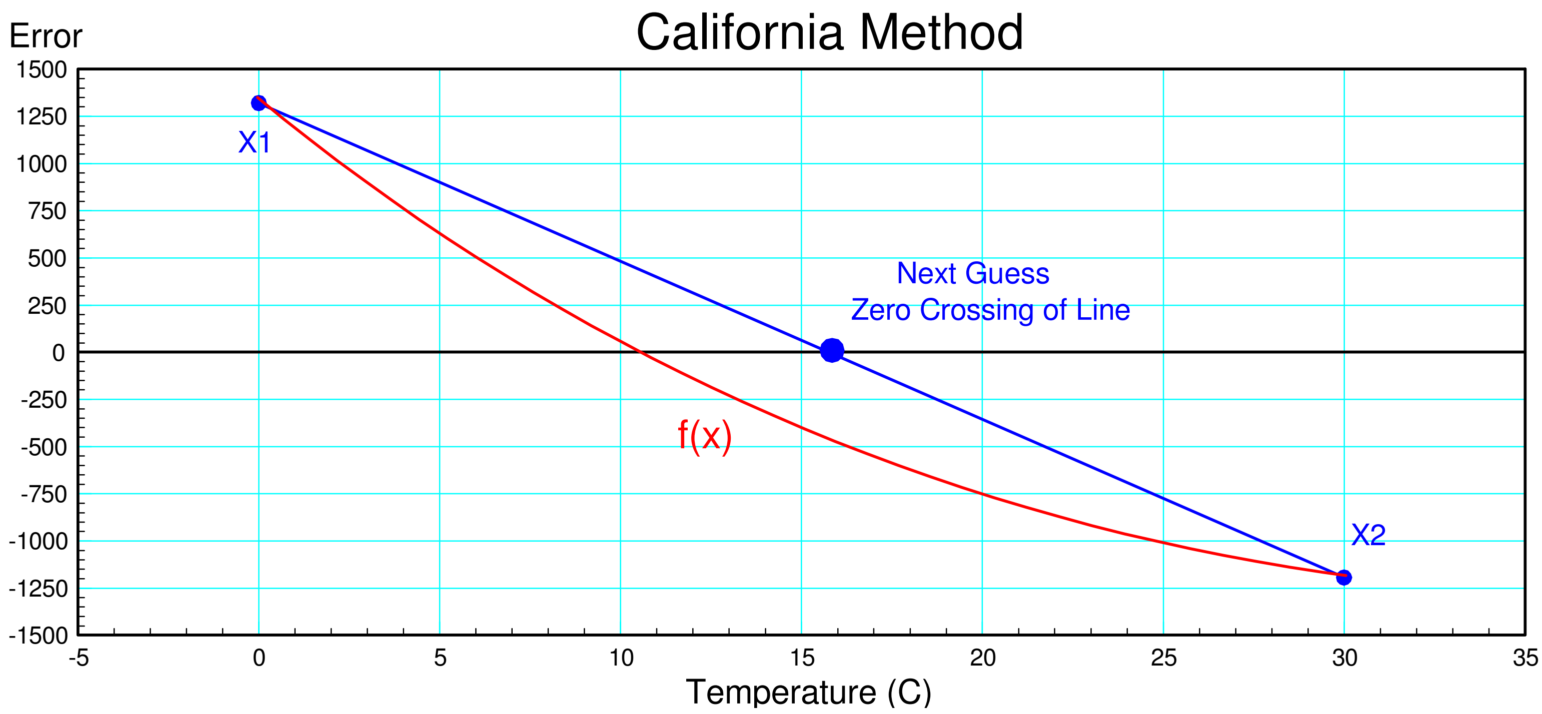
Result in the Command Window

n	T	e
1	15.0000	76.1749
2	22.5000	-382.7580
3	18.7500	-175.9167
4	16.8750	-56.1733
5	15.9375	8.3354
6	16.4063	-24.3236
7	16.1719	-8.0967
8	16.0547	0.0935
9	16.1133	-4.0080
10	16.0840	-1.9588
11	16.0693	-0.9331
12	16.0620	-0.4199
13	16.0583	-0.1632
14	16.0565	-0.0348
15	16.0556	0.0294

California Method:

- Start with two guesses (one high, one low).
- Interpolate for the next guess (rather than the midpoint)

$$X_3 = X_1 + \left(\frac{\delta X}{\delta error}\right) E_1$$



California Method in Action

- note: California method converges much faster

Matlab Script

```
X1 = 0;
Y1 = Therm(X1);
X2 = 30;
Y2 = Therm(X2);
for n=1:10
   X3 = X2-(X2-X1)/(Y2-Y1)*Y2;
   Y3 = Therm(X3);

   if(Y3 > 0)
      X1 = X3;
      Y1 = Y3;
   else
      X2 = X3;
      Y2 = Y3;
   end

   disp([n, X3, Y3]);
end
```

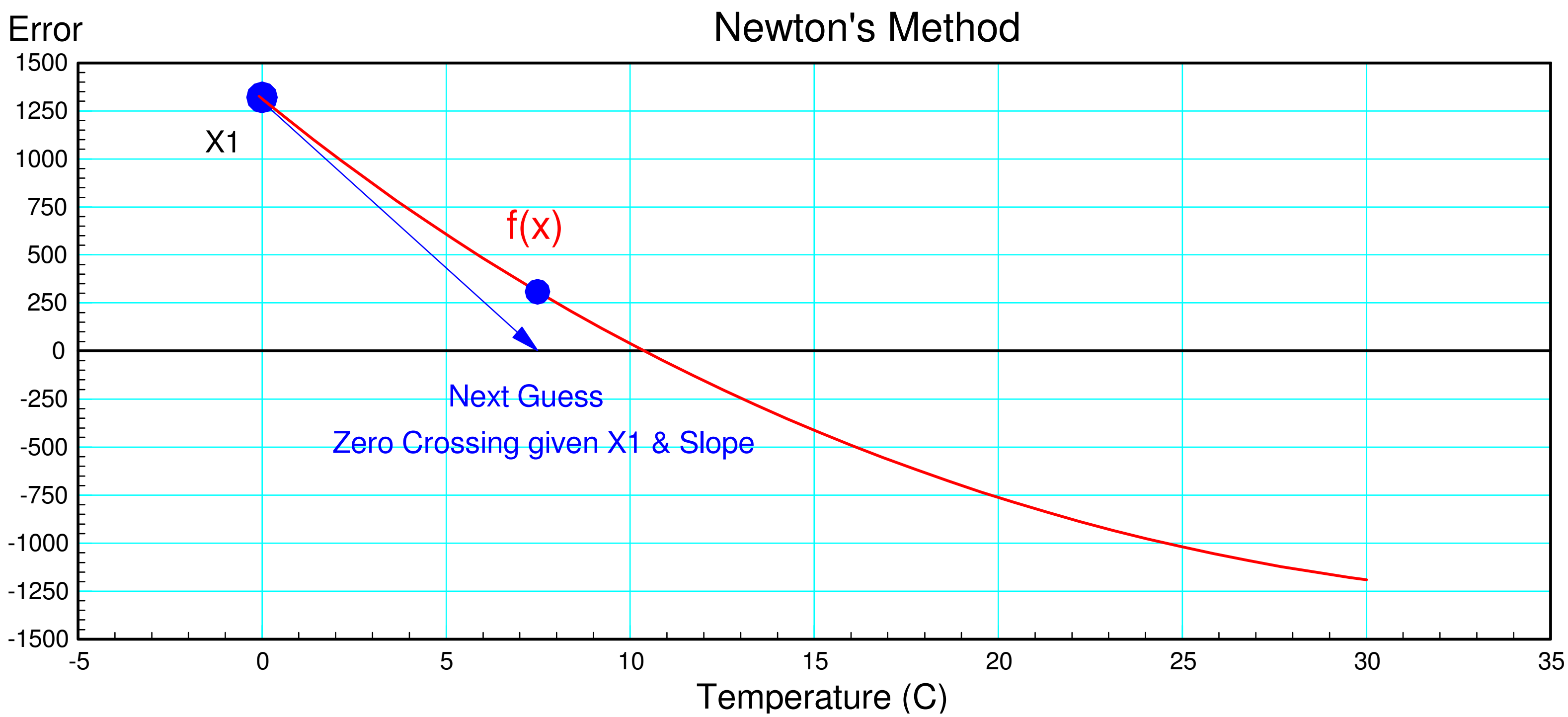
Result in the Command Window

n	T	e
1	21.7148	-342.7238
2	18.2739	-146.6308
3	16.9115	-58.6213
4	16.3838	-22.7808
5	16.1813	-8.7541
6	16.1039	-3.3494
7	16.0743	-1.2794
8	16.0630	-0.4884
9	16.0587	-0.1864
10	16.0570	-0.0711
11	16.0564	-0.0271
12	16.0562	-0.0104
13	16.0561	-0.0040
14	16.0560	-0.0015
15	16.0560	-0.0006

Newton's Method:

- Take a guess.
- Take another guess slightly larger.
- Interpolate to find the zero crossing

$$X_2 = X_1 - \left(\frac{\delta X}{\delta e}\right) e_1$$



Newton's Method in Action

- Newton's method converges very fast
- Any method with the name *Gauss* or *Newton* is probably a good method

Matlab Script

```
X3 = 0;

for n=1:10
   X1 = X3;
   Y1 = Therm(X1);
   X2 = X1 + 0.01;
   Y2 = Therm(X2);
   X3 = X2 - (X2-X1)/(Y2-Y1)*Y2;
   disp([n, X1, Y1]);
   X1 = X3;
   end
```

Result in the Command Window

n	T	e
1	0.0000	1651.2
2	11.0772	400.7303
3	15.4353	44.2472
4	16.0459	0.7072
5	16.0560	0.0000
6	16.0560	-0.0000
7	16.0560	-0.0000
8	16.0560	-0.0000
9	16.0560	-0.0000

More Fun with Newton's Method

Assume

$$R = 1000 \cdot \exp\left(\frac{3905}{T+273} - \frac{3905}{298}\right)\Omega$$

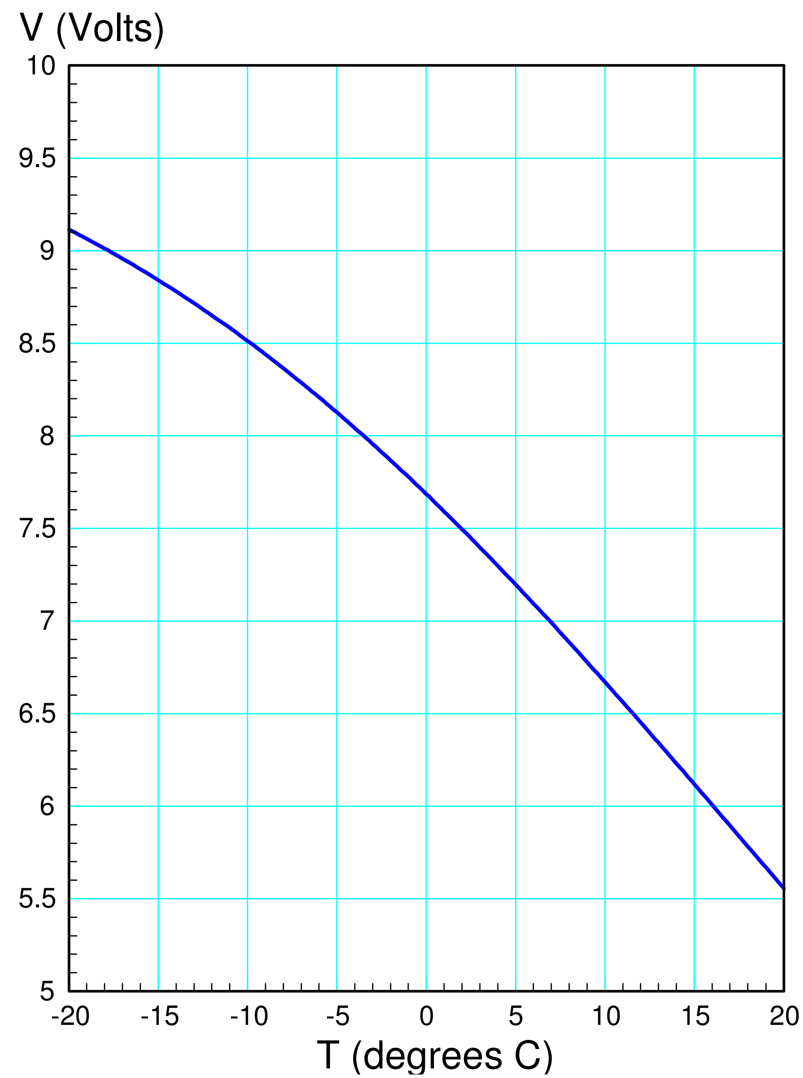
$$V = \left(\frac{R}{R+1000}\right) \cdot 10V$$

Find the temperature when

- V = 8V
- V = 7V
- V = 6V

Solution using Newton's Method:
Create a Matlab function which

- Is passed your guess at the temperature, T, and
- Returns the error in the voltage



Matlab Function

- Change V0 for each solution (8V, 7V, 6V)

```
function [e] = Voltage(T)
   V0 = 8.0;     % target voltage

   R = 1000 * exp( 3905/(T+273) - 3905/298 );
   V = R / (1000 + R) * 10;

   e = V - V0;

   end
```

Use Newton's method to solve

Matlab Script (Newton's Method)

```
X3 = 0;   % initial guess

for n=1:10
   X1 = X3;
   Y1 = Voltage(X1);
   X2 = X1 + 0.01;
   Y2 = Voltage(X2);
   X3=X2-(X2-X1)/(Y2-Y1)*Y2;

   disp([n, X1, Y1]);
   X1 = X3;
end
```

Result (V0 = 8, 7, 6)

n	T	error
1.0000	0	-0.3147
2.0000	-3.3764	-0.0115
3.0000	-3.5095	-0.0000
4.0000	-3.5098	-0.0000
5.0000	**-3.5098**	-0.0000

n	T	error
1.0000	0	0.6853
2.0000	7.3510	-0.0472
3.0000	6.9030	-0.0001
4.0000	6.9017	-0.0000
5.0000	**6.9017**	-0.0000

n	T	error
1.0000	0	1.6853
2.0000	18.0785	-0.2272
3.0000	16.0580	-0.0002
4.0000	16.0560	-0.0000
5.0000	**16.0560**	-0.0000

Newton's Method with Multiple Solutions

Your initial guess usually determines which solution it converges to

- It helps to know the answer to find the answer

Example: Find all solutions to

$$y = \frac{\cos(3x)}{x^2+1}$$

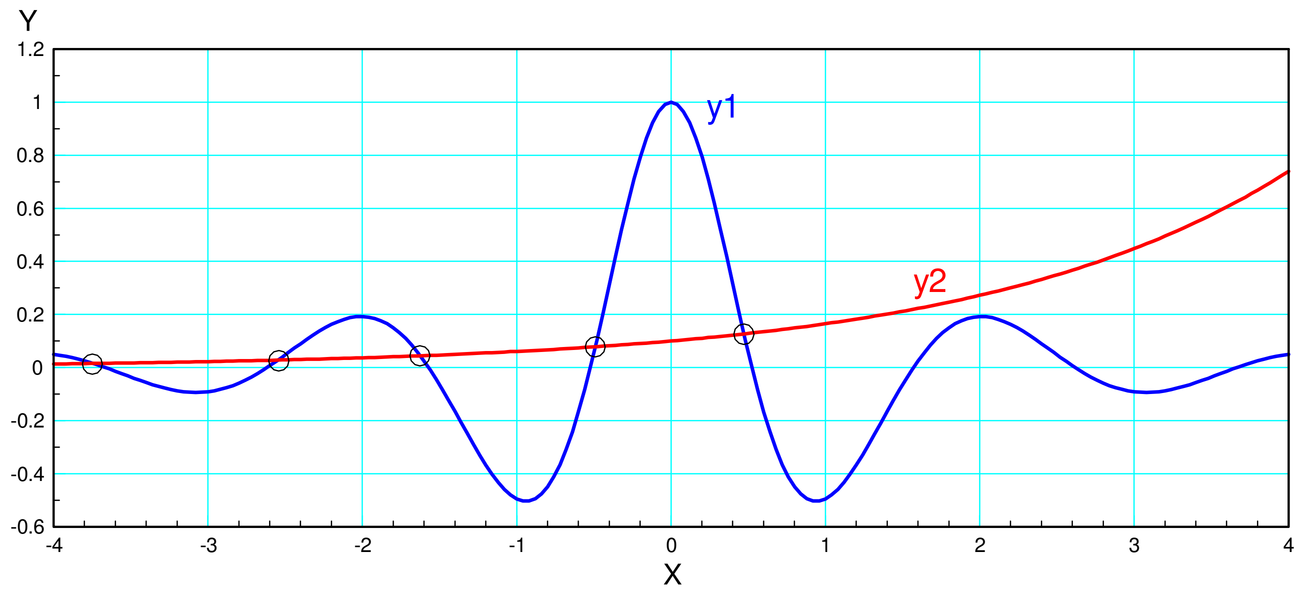
$$y = 0.1 \exp\left(\frac{x}{2}\right)$$

Method #1: Graphical Methods: Treat these as two separate functions and plot them together

$$y_1 = \frac{\cos(3x)}{x^2+1} \qquad\qquad y_2 = 0.1\exp\left(\frac{x}{2}\right)$$

The intersections are the solutions (there are five solutions)

```
>> x = [-4:0.04:4]';
>> y1 = cos(3*x) ./ (x.^2 + 1);
>> y2 = 0.1*exp(x/2);
>> plot(x,y1,x,y2)
```



Method #2: Newton's Method.

- Create a Matlab function that returns the error: y1 - y2:

```
function [e] = Example3(x)

   y1 = cos(3*x) / (x^2 + 1);
   y2 = 0.1*exp(x/2);

   e = y1 - y2;

end
```

Use Newton's method to solve.

- The initial guess pretty much determines which solution you converge to:

Matlab Script (Newton's Method)

```
X3 = -3.6;

for n=1:10
   X1 = X3;
   Y1 = Example3(X1);
   X2 = X1 + 0.01;
   Y2 = Example3(X2);
   X3=X2-(X2-X1)/(Y2-Y1)*Y2;

   disp([n, X1, Y1]);
   X1 = X3;

end
```

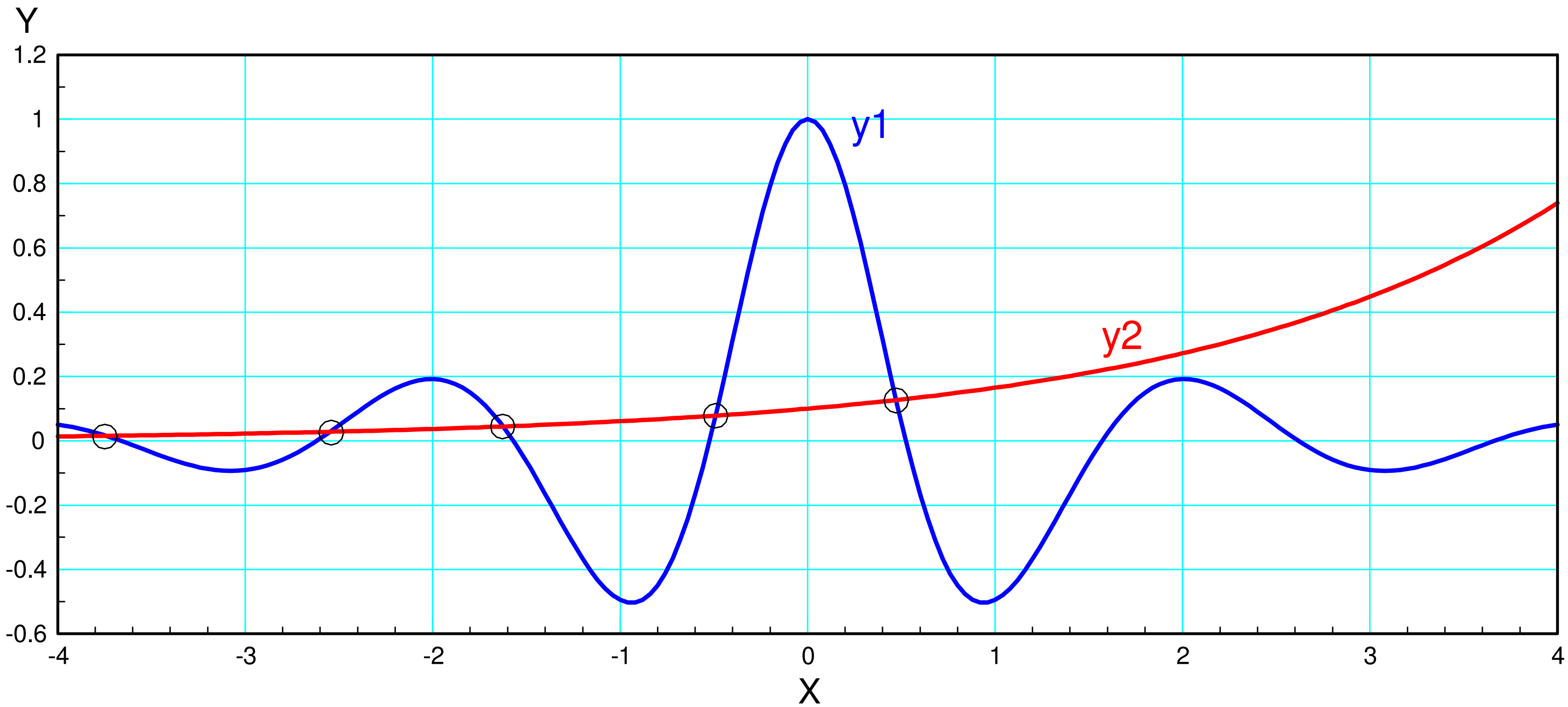
Result (Matlab command window)

n	x	y1-y2
1	-3.6000	-0.0305
2	-3.7343	-0.0017
3	-3.7428	-0.0000
4	-3.7429	-0.0000
5	**-3.7429**	-0.0000

n	x	y1-y2
1	-2.4000	0.0599
2	-2.5498	-0.0009
3	-2.5476	-0.0000
4	-2.5476	-0.0000
5	**-2.5476**	-0.0000

n	x	y1-y2
1	-1.6000	-0.0204
2	-1.6240	-0.0007
3	-1.6249	-0.0000
4	-1.6249	-0.0000
5	**-1.6249**	-0.0000

Result:



The five solutions are x = {-3.7429, -2.5476, -1.6249, -0.4912, 0.4718}

Summary:

Algebra is useful when you want to solve a mathematical equation.

You can also solve mathematical equations in Matlab using

- Graphical techniques, and
- Numeric techniques.

Methods with the name of Gauss or Newton tend to be really good methods.