Math 165: Calculus I Differentiation

ECE 111 Introduction to ECE

Week #6

Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

Differentiation

The derivative of a function gives you the slope.

This has several uses.

- Can be used to find what's happening at a given time
- Can be used to find the maximum & minimum of a function



Example: Bank Account

The derivative is the slope: this tells you what is happening with your bank account:

0<x<2: The slope is zero

• Nothing is being added or taken out

2<x<4: The slope is positive

• Money is being added to your account

5<x<6: Negative slope

• Money is being withdrawn

Furthermore, the derivative also tells you how much money you're depositing or withdrawing.



Calculating the Derivative

Determine the slope:

$$slope = y'(x) = \frac{dy}{dx} = \frac{\text{change in y}}{\text{change in x}}$$

0<x<2

• dy = 0, dx = 2, dy/dx = 0

2<x<4

• dy = 20, dx = 2, dy/dx = 10

7<x<9

• dy = +30, dx = 2, dy/dx = 15



Example: Global CO2 Levels

• 2015 - 2023

Graph shows global CO2 levels

Derivative shows amount of CO2 added

- Positive Slope: CO2 is being added
- Negative Slope: CO2 is bring removed



What's Happening?

Derivatives are useful

• They provide information about what's happening on a month-by-month basis.

Most land is in the northern hemisphere

Summer (July)

- Global CO2 levels drop.
- Trees green up and absorb CO2

Winter (January)

- Global CO2 levels rise
- Trees drop their leaves
- Leaves decay, releasing CO2

Also shows a hiccup in March

• Not sure why...



Topics for This Lecture

How to find a derivative

- Using lookup tables similar to what you'll do in Math 165,
- Using graphical methods, and
- Using numerical methods (i.e. Matlab).

Evaluate different paths for a robotic arm

- 1st derivative of angle = velocity
 - Velocity = voltage for a DC motor
- 2nd derivative of angle = acceleration
 - Acceleration = current for a DC motor

Different paths result in different implied voltages and currents



Differentiation: Method #1

Memorize and apply a set of rules.

- What you do in Calculus I
- **Power Functions:**

Exponential

Sine Function

Cosine Function

Chain Rule

Division

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

$$\frac{d}{dx}(e^{ax}) = a \cdot e^{ax}$$

$$\frac{d}{dx}(\sin(ax)) = a \cdot \cos(ax)$$

$$\frac{d}{dx}(\cos(ax)) = -a \cdot \sin(ax)$$

$$\frac{d}{dx}(f(x) \cdot g(x)) = \frac{df(x)}{dx} \cdot g(x) + f(x) \cdot \frac{dg(x)}{dx}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot \frac{df}{dx} - f(x) \cdot \frac{dg}{dx}}{(g(x))^2}$$

With these rules, you can find

$$y = \frac{d}{dx} \left(\frac{\cos(x)}{x^2 + 1} \right)$$

Stay tuned until you take Calculus I to solve using this method.

Differentiation Method #2: Graphical

- Draw the tangent at several points
- Determine the slope at each point
- Draw a curve connecting the slopes

Example: Find the derivative of $y = x^2$



Step 1: Draw the tangent at several points (shown in red)

Step 2: Determine the slope at each point (shown in purple)



Step 3: Connect the points (shown in purple)



Example #2: Determine the derivative of a sine wave:

 $y = \sin(x)$

- Step 1: Plot y vs. x (shown below)
- Step 2: Compute the slope at several points



Step 3: Plot the slopes and connect them with a smooth curve



The result (shown in red) is a cosine function

Graphical Example #3:

• Works for any function

Example: $y = \frac{d}{dx} \left(\frac{\cos(x)}{x^2 + 1} \right)$



Step 1: Compute the slope at several points



Step 2: Plot the derivatives and connect with a smooth curve



Differentiation Method #3: Numerical Differentiation.

Repeating, the derivative is

 $\frac{dy}{dx} \equiv \frac{\text{the change in y}}{\text{the change in x}}$

You can approximate this as

$$\left(\frac{dy}{dx}\right)_{x=4} \approx \frac{y(5) - y(3)}{x(5) - x(3)}$$



Repeat a a bunch of points

 $y'(i) \approx \frac{y(i+1)-y(i-1)}{x(i+1)-x(i-1)}$

Note: Endpoints cause problems

• Can't go beyond the endpoints

Use a slightly different equation for the endpoints

$$y(1) \approx \frac{y(2) - y(1)}{x(2) - x(1)}$$

 $y'(n) \approx \frac{y(n) - y(n-1)}{x(n) - x(n-1)}$

Differentiation in Matlab

Create a function

- A Matlab routine you can call
- Similar to cos(x) or exp(x)

In Matlab, click on 'File New Function'



Creating A New Function in Matlab: File New Function

In the editor window, type in the following:

```
function [dy ] = derivative( x, y )
npt = length(x);
dy = 0*x;
for i=2:npt-1
    dy(i) = ( y(i+1) - y(i-1) ) / ( x(i+1) - x(i-1) );
end
dy(1) = (y(2) - y(1)) / ( x(2) - x(1) );
dy(npt) = (y(npt) - y(npt-1)) / ( x(npt) - x(npt-1));
end
```

Save as 'deravitive.m'

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Check your function

- Always a good idea
- Check against a known answer

From Math 165,

 $\frac{d}{dx}(2\sin(3t)) = 6\cos(3t)$

Checking in Matlab:

```
>> x = [0:0.1:4]';
>> y = 2*sin(3*x);
>> dy = derivative(x,y);
>> plot(x,dy,'b.',x,6*cos(3*x));
```

The two match up

- The derivative function seems to be working
- The endpoints are a little off (common problem)



Example 2:

• Numerical solutions work even if you don't know the answer

Find the derivative of

 $y = e^{-x^2} \cdot \cos(x^3)$

First, input y(x) into Matlab

Now find the derivative: dy/dx

>> dy = derivative(x,y);
>> plot(x,y,'b',x,dy,'r')



Finding local maximums & minimums

y(x) is a max/min when y'(x) = 0

This is how you'll find the max/min of a funciton in Math 165

- Take the derivative
- Set y'(x) = 0



Differentiation and Noise

Differentiation amplifies noise

• Major problem

Arctic Sea Ice (top curve)

- Decent data
- Seems like a downward trend

Derivative (bottom curve)

- Lots of noise
- Can't tell what's happening





Example 2:

Global Temperatures (top curve)

- Decent data
- Seems to be an upward trend



Derivative (bottom curve)

- Looks like random noise
- Can't see the trend



Robotics & Path Planning

Another application of derivatives. Problem:

- Find a path from point A to B
- Keep the 1st and 2nd derivatives small

Why?

- y(x) = Motor angle
- 1st derivative: Motor velocity
 - Roughly same as voltage for a DC motor
- 2nd derivative: Motor acceleration
 - Roughly same as current for a DC motor



Option 1: Linear Motion.

Assume motor follows the path

$$y = \begin{cases} 0 & t < 0 \\ t/2 & 0 < t < 2 \\ 1 & t > 2 \end{cases}$$

Advantage:

• Simple function

Disadvantage:

- Jump discontinuity in voltage (y')
- Current goes to infinity (y")



Find the derivatives in Matlab



Option 2: Cosine Motion

Let

$$y = \begin{cases} 0 & t < 0\\ \left(\frac{1}{2}\right) \left(1 - \cos\left(\frac{\pi t}{2}\right)\right) & 0 < t < 2\\ 1 & t > 2 \end{cases}$$

A little more complicated function

- 1st derivative is finite
- 2nd derivative is finite



In Matlab:



Other paths from A to B can be defined

- Keep acceleration constant
- Start and end with zero acceleration
- Cubic function for y(x)
- Other

This is kind of the idea with path planning

- What's the best path for going from point A to point B?
- Derivatives are used to evaluate each path

Summary:

Derivatives and differentiation are useful:

- Bank Account
 - Derivative tells you how much money is being deposited or withdrawn
- Arctic Sea Ice
 - How much ice is being added (positive derivative
 - How much ice is being lost (negative derivative)
- DC Motors
 - Voltage to the motor (1st derivative)
 - Current to the motor (2nd derivative)

Graphical methods work for any function

• The derivative is the slope at any point

Numeric methods work

- If you can get the function into Matlab, you can find the derivative
- Noise causes problems: differentiation amplifies noise