ECE 311: Circuits II

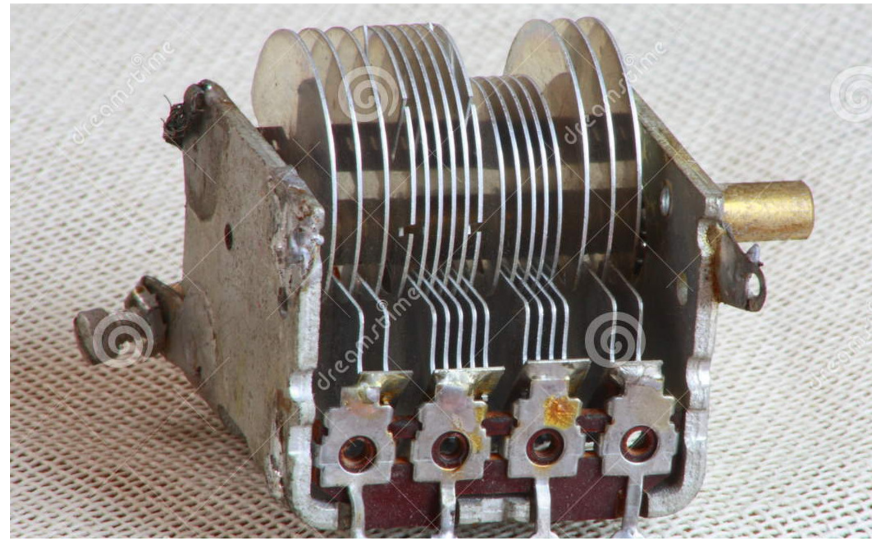
Circuits with Capacitors & The Heat Equation

ECE 111 Introduction to ECE

Week #10

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Capacitors



A capacitor is a set of parallel plates[1]

$$C = \varepsilon \frac{A}{d} \text{ (Farads)}$$

where

- ε is the dielectric constant
- (air = $8.84 \cdot 10^{-12}$)
- A is the area of the capacitor, and
- d is the distance between plates.

The area you need for 1 Farad with plates 1mm apart is

$$1F = (8.84 \cdot 10^{-12})\frac{A}{0.001m}$$

$$A = 113,122,171m^2$$

1 http://www.electronics-tutorials.ws/

Capacitors (cont'd)

The charge stored is

$$Q = C\,V$$

- 1 Coulumb = $6.242 \cdot 10^{18}$ electrons

Current is Coulumbs / Second

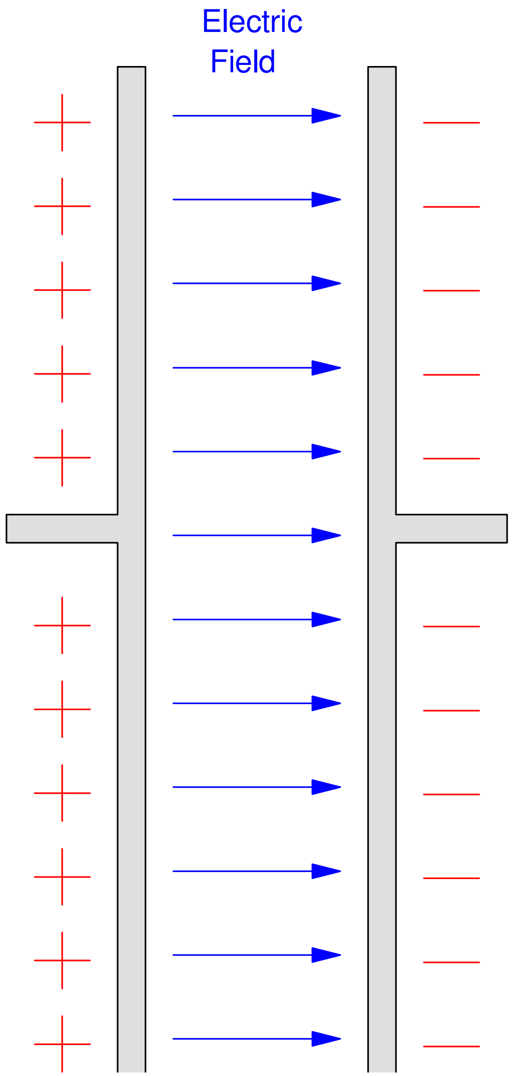
$$I = \frac{dQ}{dt} = C\frac{dV}{dt} + V\frac{dC}{dt}$$

If C = constant

$$\boxed{I = C\frac{dV}{dt}}$$

and

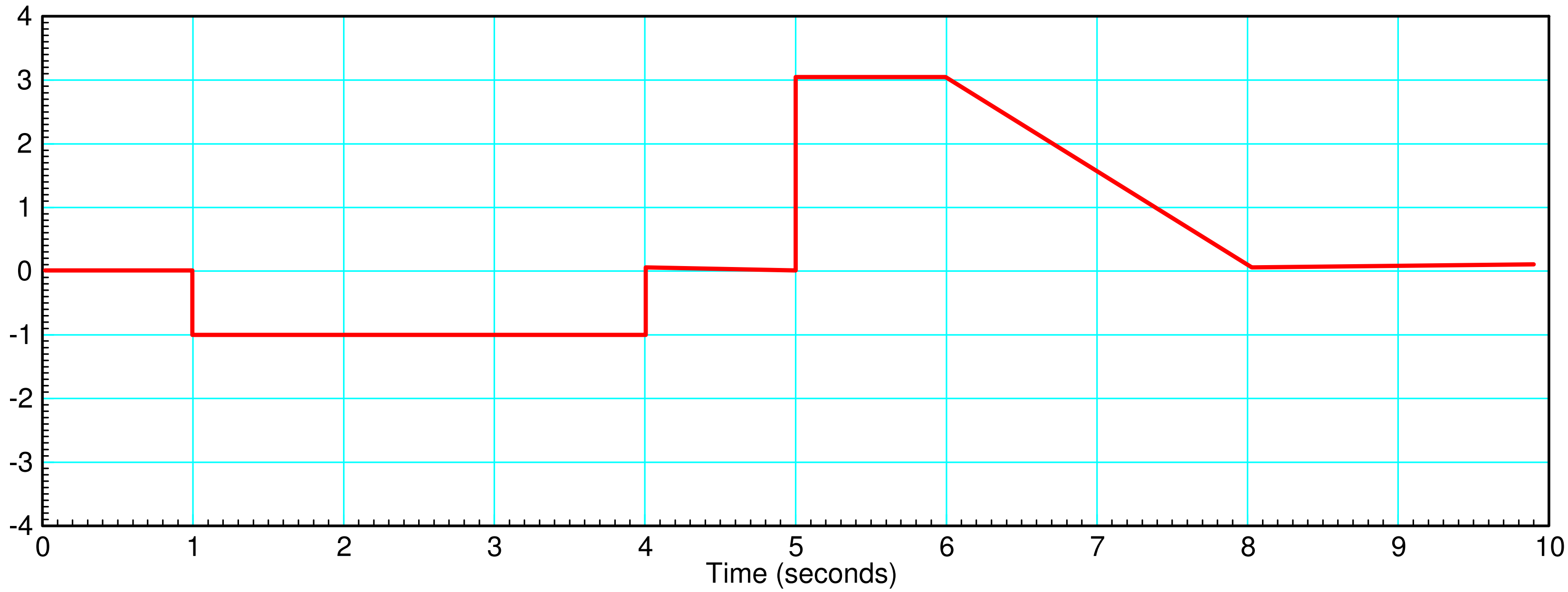
$$V = \frac{1}{C}\int I \cdot dt$$



Practice Problem:

Assume the current flowing into a 1F capacitor is as follows.

- Determine the voltage
- $V = \frac{1}{C}\int I \cdot dt$



Differential Equations and Circuits

- Each capacitor adds a 1st-order differential eqution
- A circuit with 3 capacitors is described by a 3rd-order differential equation

Any circuit with capacitors (or inductors - next week) is described by differential equations

- Hence the reason you're taking 4 semesters of calculus

In Calculus, you will be covering integration and differentiation and how to come up with a closed-form solution to various problems.

In this class (ECE 111), we will be using Matlab to solve using numerical methods

Time Response of an RC filter: (Heat Equation)

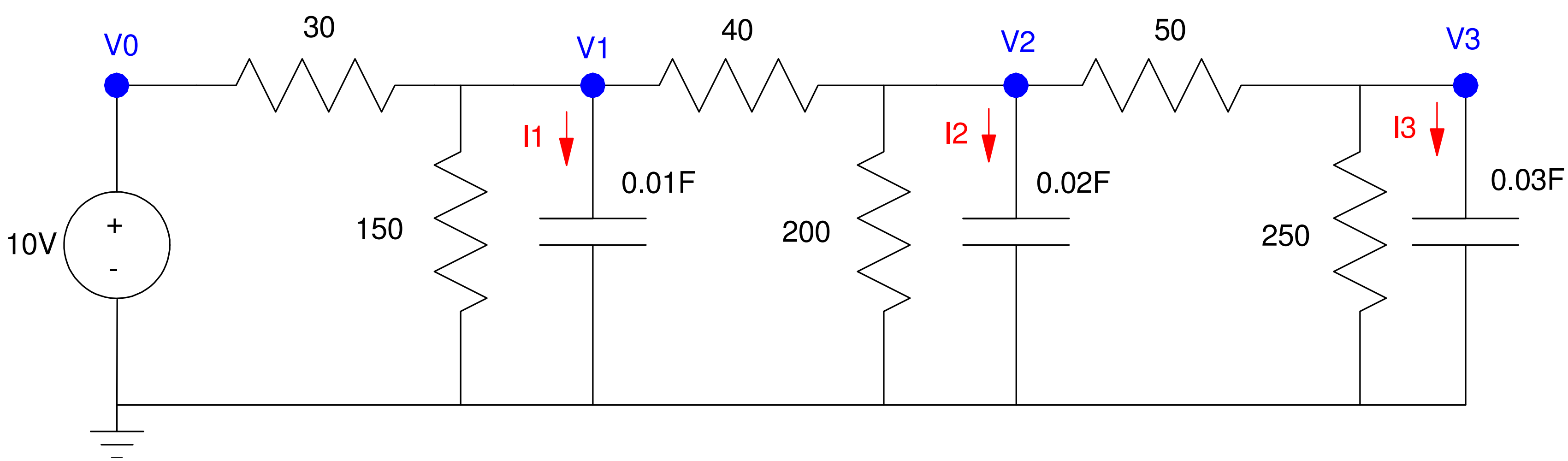
Add a capacitor to each node from last week's circuit

- Produces a 3rd-order differential eqation

At steady state

- $V_i =$ constant
- $\frac{dV_i}{dt} = 0$

You get the solution from last week

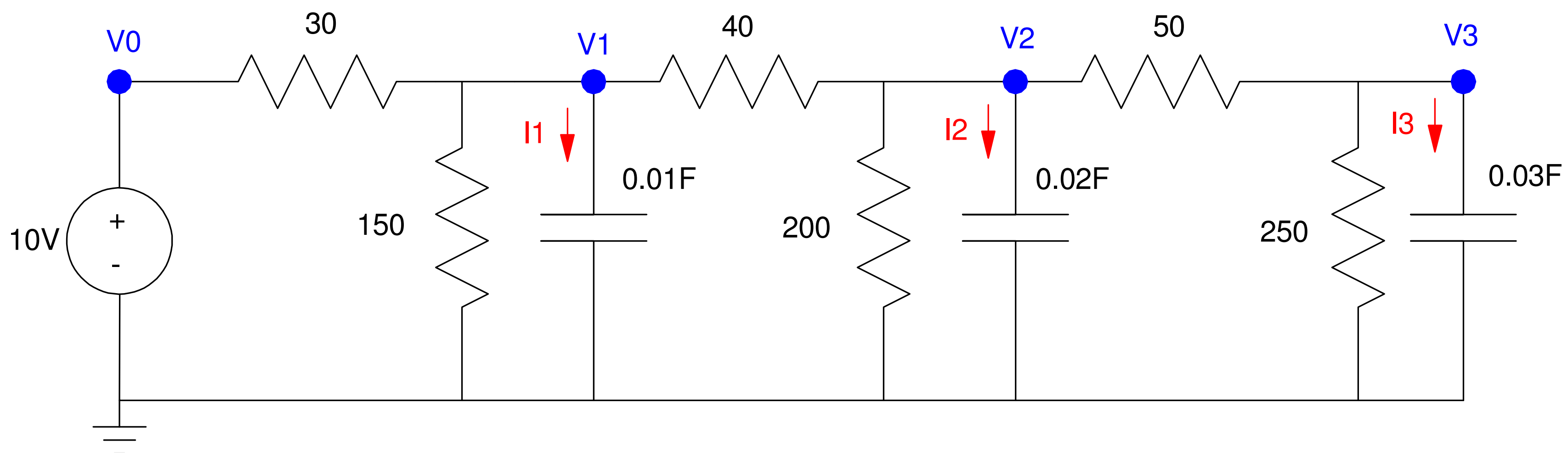


Transient Response

Assume V1(0) = V2(0) = V3(0) = 0

For t > 0

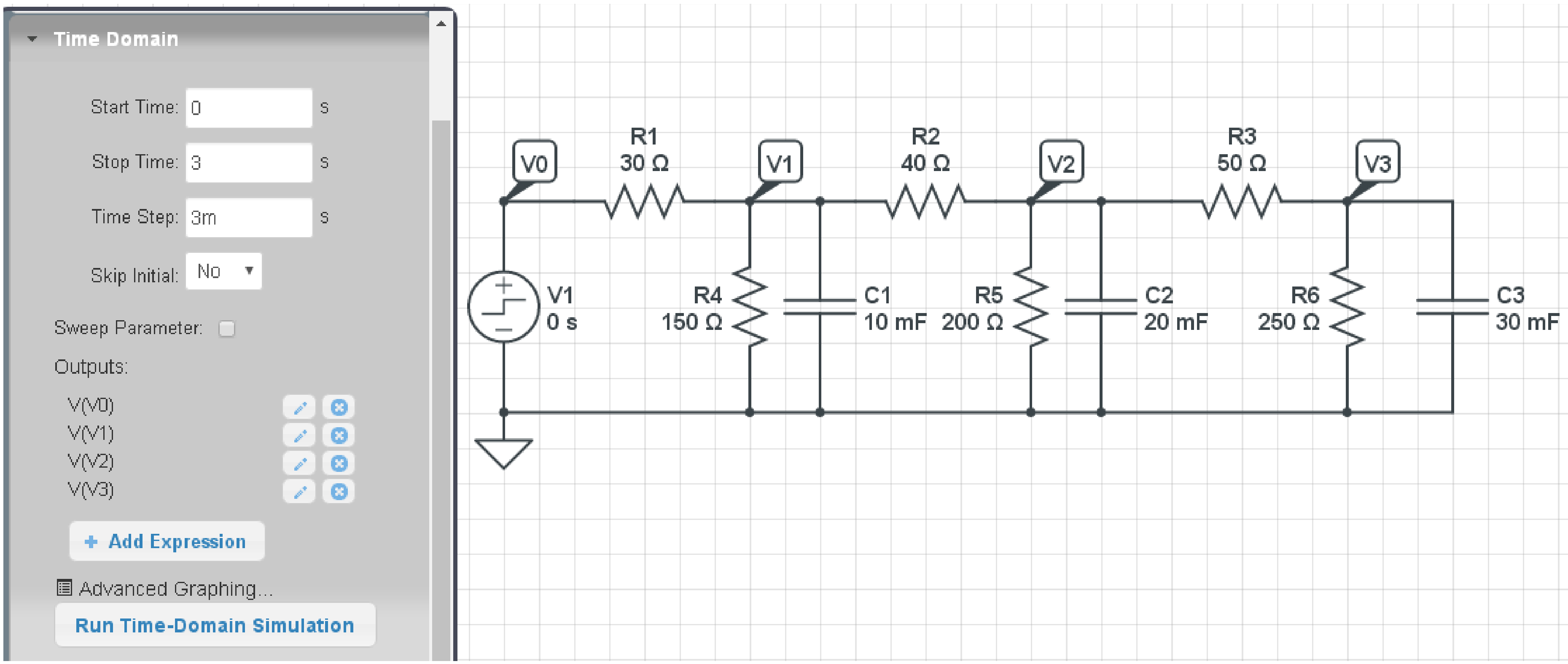
- Current starts to flow into C1
- V1 starts to increase
- Which then charges up C2
- Which then charges up C3



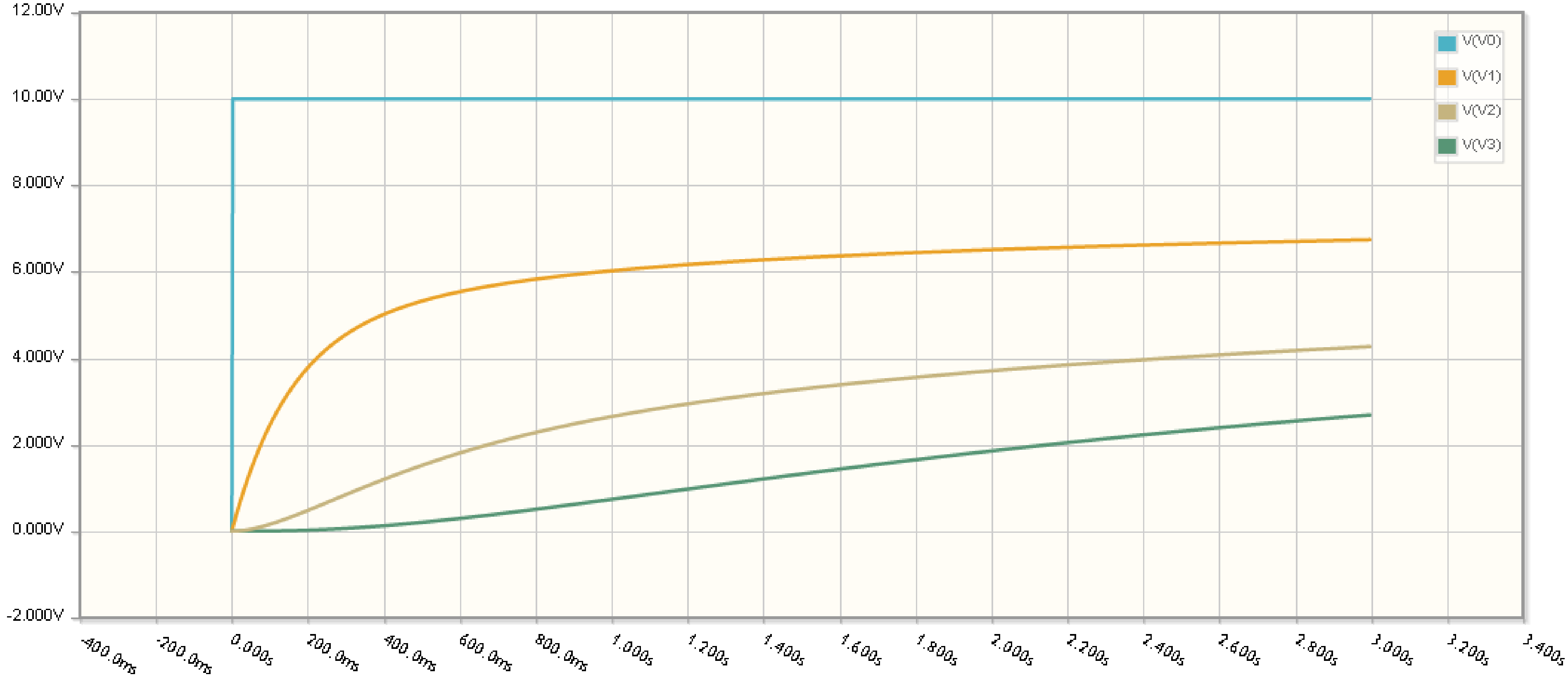
CircuitLab Simulation

This show up in the CircuitLab simulation

Click on Run Simulation and select Transient Response



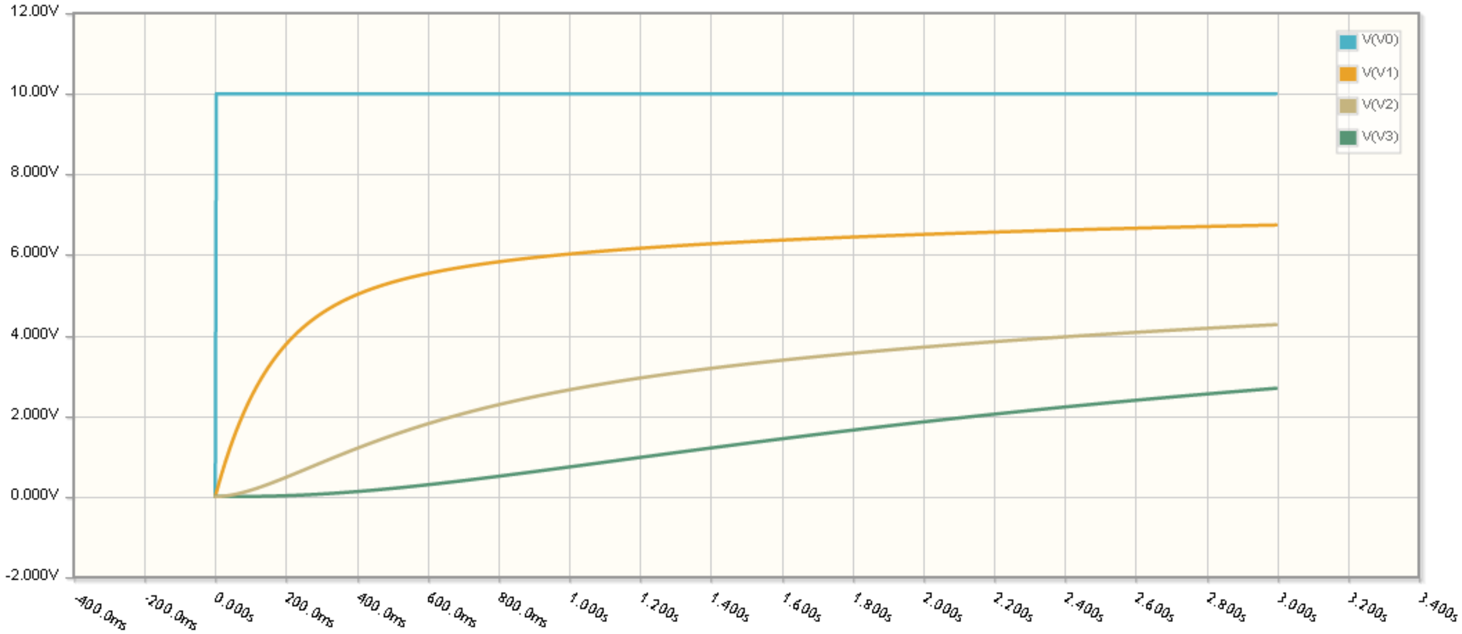
This will show you how the voltages change over time:



Transient voltages on V0, V1, V2, and V3: The capacitors are charging up to their steady-state value

What's happening is this:

- Initially, the capacitors are discharged (V = 0 at t = 0)
- When the input turns on to 10V, a current imbalance results in current flowing into the capacitors, charging them up.
- Eventually, you reach equilibrium. At this point, the current in equals the current out and no excess current remains to charge up the capacitors. At this point, you're at the steady-state solution we found last week.



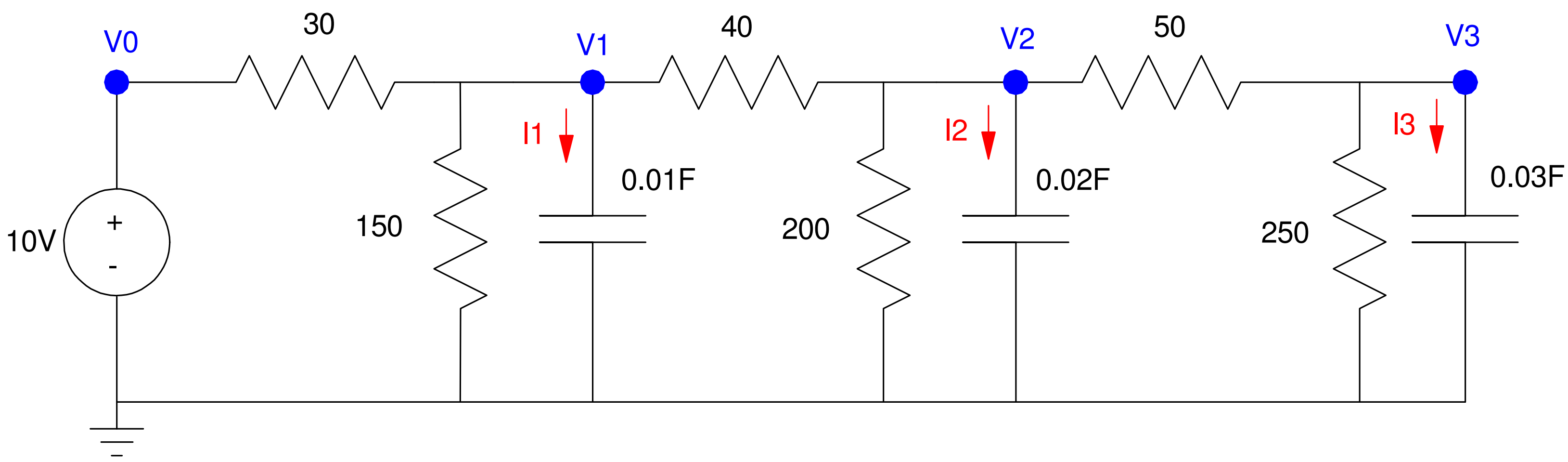
Matlab Computations

First compute the currents I1, I2, and I3 (current out = current in)

$$I_1 = \left(\frac{V_0 - V_1}{30}\right) + \left(\frac{0 - V_1}{150}\right) + \left(\frac{V_2 - V_1}{40}\right)$$

$$I_2 = \left(\frac{V_1 - V_2}{40}\right) + \left(\frac{0 - V_2}{200}\right) + \left(\frac{V_3 - V_2}{50}\right)$$

$$I_3 = \left(\frac{V_2 - V_3}{50}\right) + \left(\frac{0 - V_3}{250}\right)$$

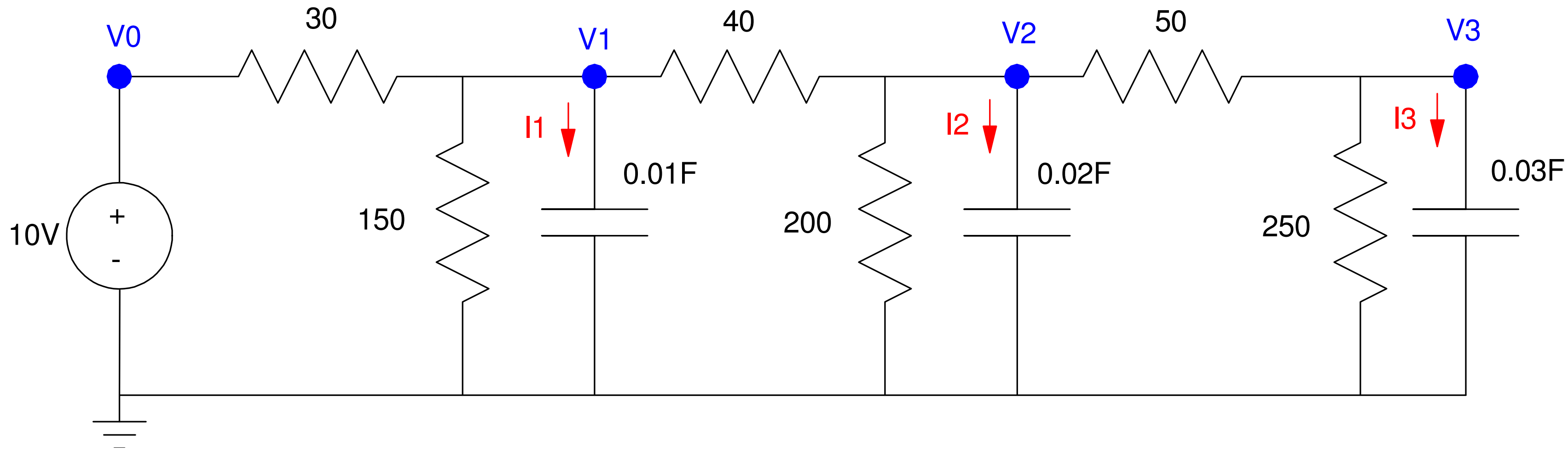


Note that the current is equal to $C\frac{dV}{dt}$

$$0.01\frac{dV_1}{dt} = I_1 = \left(\frac{V_0 - V_1}{30}\right) + \left(\frac{0 - V_1}{150}\right) + \left(\frac{V_2 - V_1}{40}\right)$$

$$0.02\frac{dV_2}{dt} = I_2 = \left(\frac{V_1 - V_2}{40}\right) + \left(\frac{0 - V_2}{200}\right) + \left(\frac{V_3 - V_2}{50}\right)$$

$$0.03\frac{dV_3}{dt} = I_3 = \left(\frac{V_2 - V_3}{50}\right) + \left(\frac{0 - V_3}{250}\right)$$



Solve for $\frac{dV_i}{dt}$

$$\frac{dV_1}{dt} = 3.333V_0 - 6.500V_1 + 2.500V_2$$

$$\frac{dV_2}{dt} = 1.250V_1 - 2.500V_2 + 1.000V_3$$

$$\frac{dV_3}{dt} = 0.667V_2 - 0.800V_3$$

Integrate to find V1..V3

$$V_1(t) = \int_0^t \frac{dV_1}{dt} \cdot d\tau$$

$$V_2(t) = \int_0^t \frac{dV_2}{dt} \cdot d\tau$$

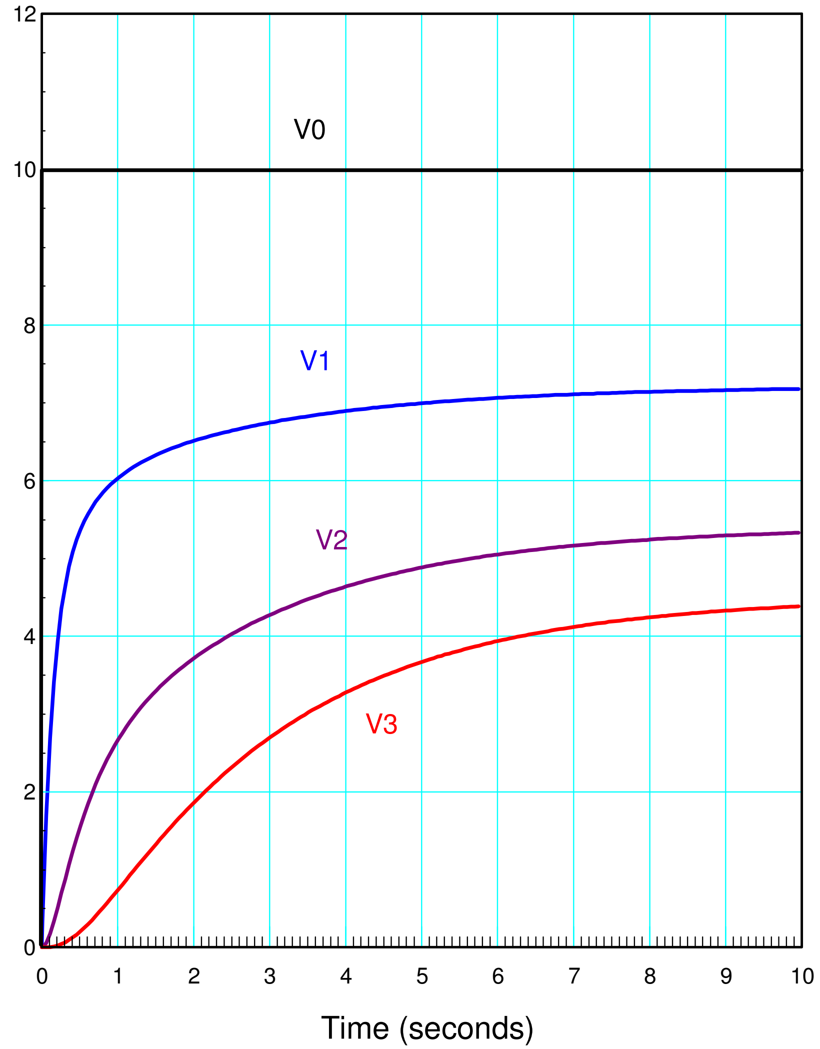
$$V_3(t) = \int_0^t \frac{dV_3}{dt} \cdot d\tau$$

```
t = 0;
dt = 0.01;
V0 = 100;
V1 = 0;
V2 = 0;
V3 = 0;

% Compute dV/dt
dV1 =  3.333*V0 - 6.500*V1 + 2.500*V2;
dV2 =  1.250*V1 - 2.500*V2 + 1.000*V3;
dV3 =  0.667*V2 - 0.800*V3;

% Integrate
 V1 = V1 + dV1*dt;
 V2 = V2 + dV2*dt;
 V3 = V3 + dV3*dt;
```

Repeat 1000 times

- 10 seconds

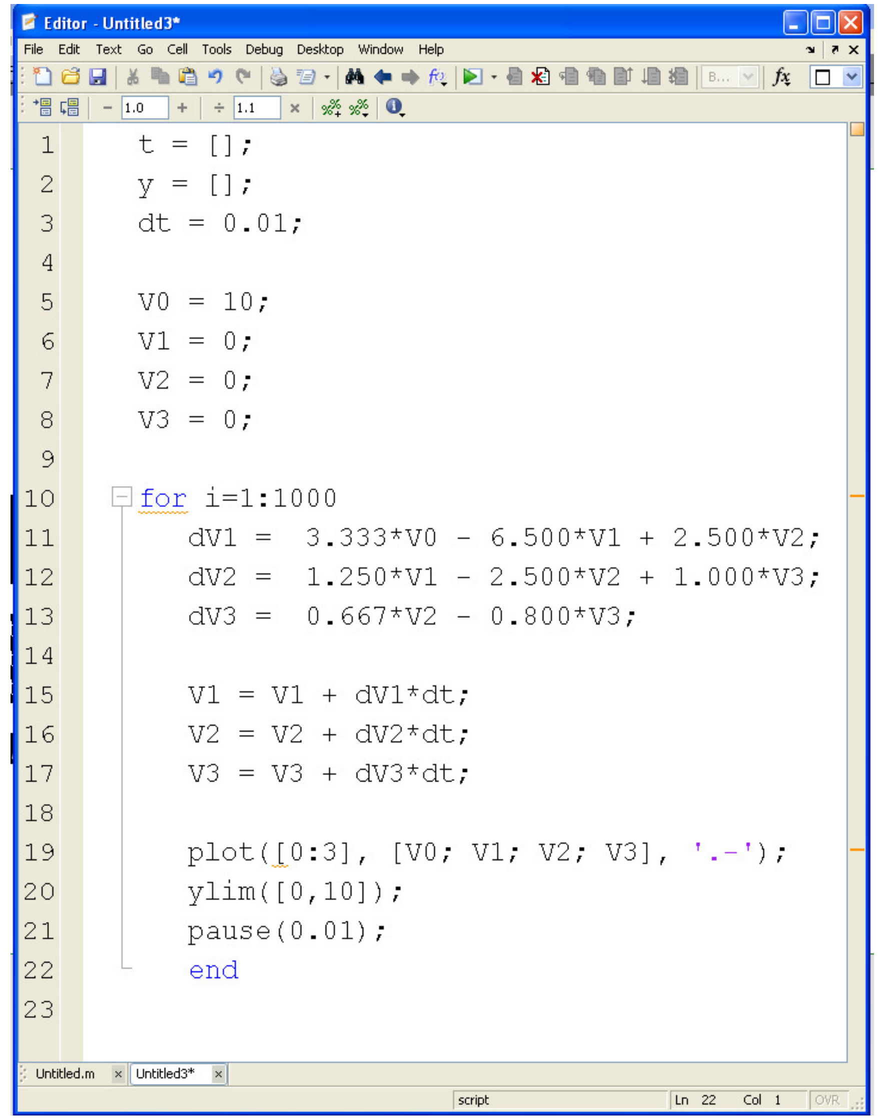
```
t = [];
y = [];
dt = 0.01;
V0 = 10;
V1 = 0;
V2 = 0;
V3 = 0;
for i=1:1000
   dV1 =  3.333*V0 - 6.500*V1 + 2.500*V2;
   dV2 =  1.250*V1 - 2.500*V2 + 1.000*V3;
   dV3 =  0.667*V2 - 0.800*V3;
   V1 = V1 + dV1*dt;
   V2 = V2 + dV2*dt;
   V3 = V3 + dV3*dt;
   y = [y; V1, V2, V3];
   end
t = [1:1000]' * dt;
plot(t,y);
xlabel('Time (seconds)');
ylabel('V(t)');
```



Animation in MATLAB

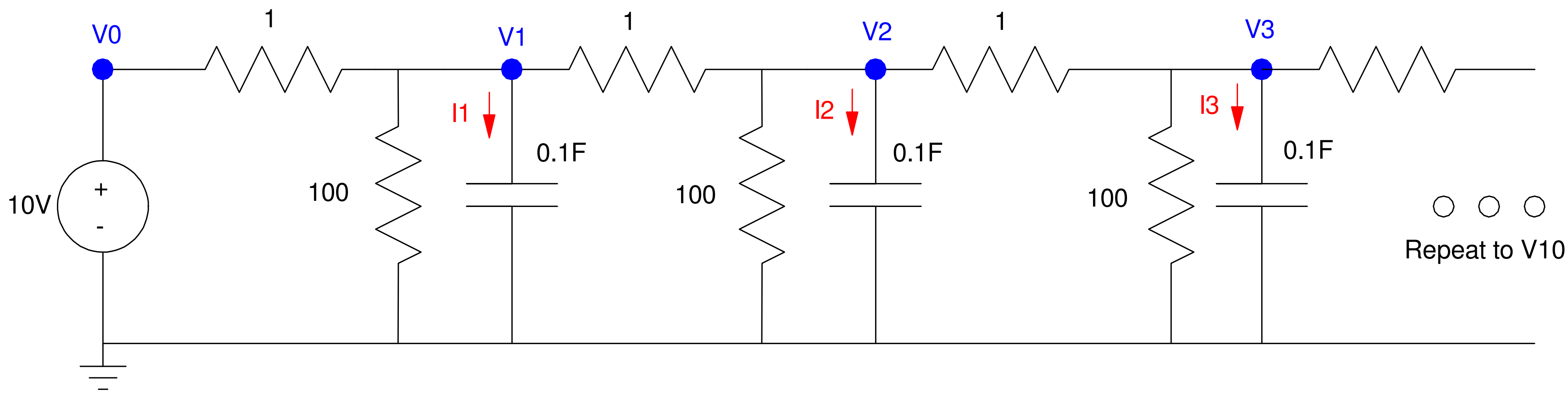
You can also watch the voltages change vs. time. The trick is

- Plot your function (the node voltages in this case), and
- Insert a *pause(0.01)* command to pause the MATLAB program and display the current temperature



Animation of a 10-Stage RC Filter

Take the 10-stage RC filter from last week and add a 0.1F capacitor to each node:

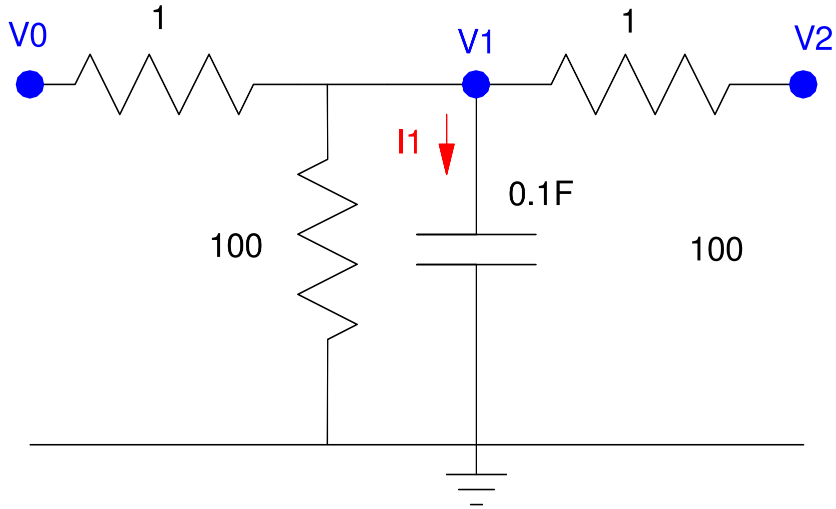


10-Stage RC Filter. R and C for each stage are the same.

10-Stage RC Filter:

Write the differential equation at each node

$$I_{C1} = C_1 \frac{dV_1}{dt} = \left(\frac{V_0 - V_1}{1}\right) + \left(\frac{V_2 - V_1}{1}\right) + \left(\frac{0 - V_1}{100}\right)$$



Simplify

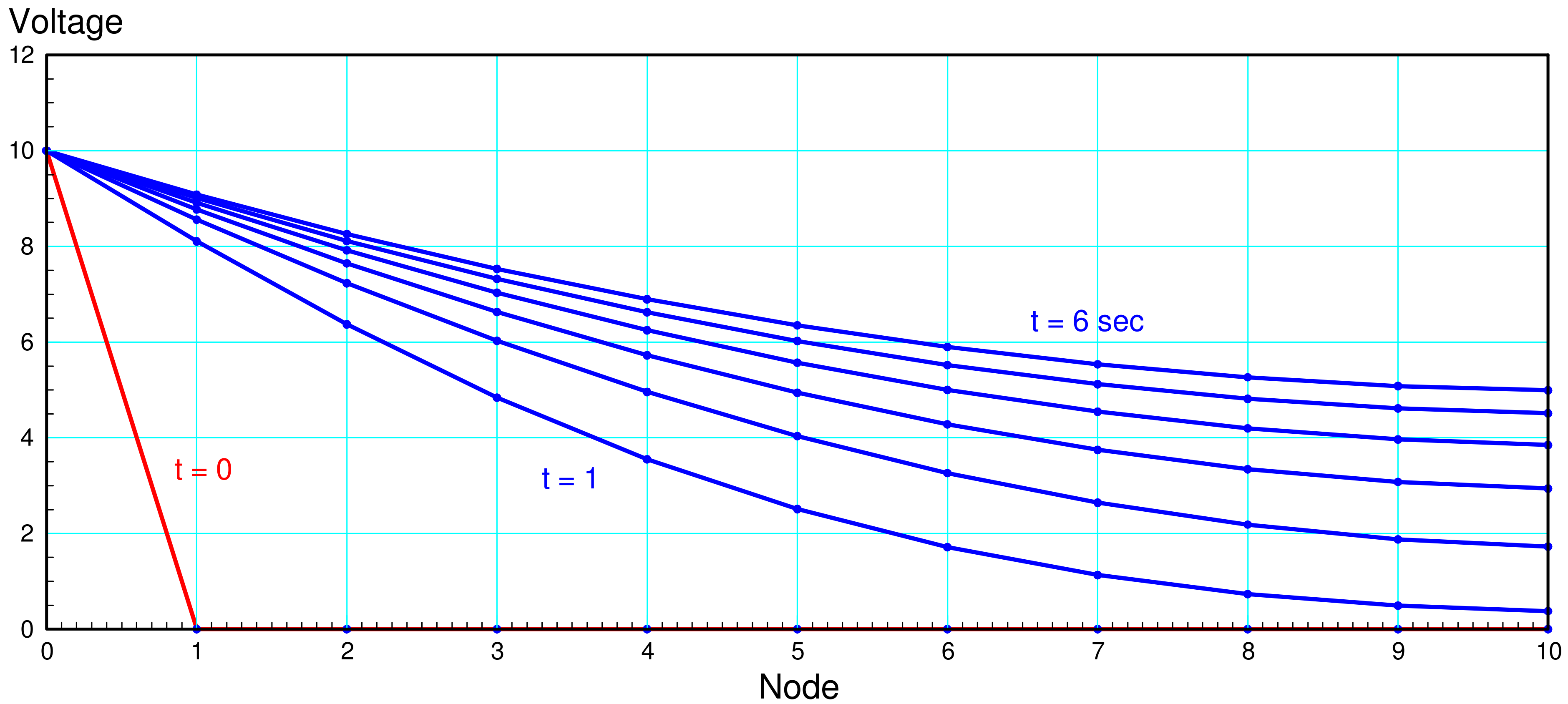
$$\frac{dV_1}{dt} = 10V_0 - 20.1V_1 + 10V_2$$

Ditto for nodes 2..9. Node #10 is different

$$I_{C_{10}} = C_{10} \frac{dV_{10}}{dt} = \left(\frac{V_9 - V_{10}}{1}\right) + \left(\frac{0 - V_{10}}{100}\right)$$

$$\frac{dV_{10}}{dt} = 10V_9 - 10.1V_{10}$$

```
V = zeros(10,1);
dV = zeros(10,1);
V0 = 10;
dt = 0.01;
t = 0;

while(t < 100)

   dV(1)  = 10*V0   - 20.1*V(1) + 10*V(2);
   dV(2)  = 10*V(1) - 20.1*V(2) + 10*V(3);
   dV(3)  = 10*V(2) - 20.1*V(3) + 10*V(4);
   dV(4)  = 10*V(3) - 20.1*V(4) + 10*V(5);
   dV(5)  = 10*V(4) - 20.1*V(5) + 10*V(6);
   dV(6)  = 10*V(5) - 20.1*V(6) + 10*V(7);
   dV(7)  = 10*V(6) - 20.1*V(7) + 10*V(8);
   dV(8)  = 10*V(7) - 20.1*V(8) + 10*V(9);
   dV(9)  = 10*V(8) - 20.1*V(9) + 10*V(10);
   dV(10) = 10*V(9) - 10.1*V(10);

   V = V +  dV*dt;
   t = t + dt;

   plot([0:10], [V0;V], '.-');
   ylim([0,10]);
   pause(0.01);
   end
```



Temperature Along a Bar plotted every 1.00 second

Eigenvalues and Eigenvectors

Suppose you want to solve the differential equation

$$\frac{dx}{dt} = -3x$$

with

$$x(0) = x_0.$$

In Math 166, you assume x(t) is in the form of

$$x(t) = e^{st}.$$

Then

$$\frac{dx}{dt} = s \cdot e^{st} = sx$$

Substituting into the above differential equation results in

$$sx = -3x$$

$$(s+3)x = 0$$

Either

- x(t) = 0 (the trivial solution), or
- s = -3

This means x(t) is in the form of

$$x(t) = a \cdot e^{-3t}$$

Plug in the initial conditions and you get

$$x(t) = x_0 \cdot e^{-3t}$$

This also works for matrices. If

$$\dot{X} = AX$$

then

$$X(t) = e^{At} X_0$$

or in terms of eigenvalues and eigenvectors

$$X(t) = a_1 \Lambda_1 e^{\lambda_1 t} + a_2 \Lambda_2 e^{\lambda_2 t} + \ldots a_{10} \Lambda_{10} e^{\lambda_{10} t}$$

where

- Λ_i is the ith eigenvector,
- λ_i is the ith eigenvalue, and
- a_i are constants determined by the initial condition.

Eigenvalues tell you how the system behaves

Eigenvectors tell you what behaves that way.

$$X(t) = a_1 \Lambda_1 e^{\lambda_1 t} + a_2 \Lambda_2 e^{\lambda_2 t} + \ldots a_{10} \Lambda_{10} e^{\lambda_{10} t}$$

If X(0) is equal to an eigenvector, then only that one mode is excited.

- The shape of x(t) remains the same (only one eigenvector is excited)
- x(t) then goes to zero according to its eigenvalue.

If

$$X(0) = \Lambda_2$$

then

$$X(t) = \Lambda_2 e^{\lambda_2 t}$$

If X(0) excites multiple eigenvectors, then X(t) will be the combination of all its eigenmodes.

Example: Take for example, the 10-stage RC filter. In matrix form

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \\ \dot{V}_5 \\ \dot{V}_6 \\ \dot{V}_7 \\ \dot{V}_8 \\ \dot{V}_9 \\ \dot{V}_{10} \end{bmatrix} = \begin{bmatrix} -20.1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & -20.1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & -20.1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & -20.1 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & -20.1 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & -20.1 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & -20.1 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & -20.1 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & -20.1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & -10.1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \\ V_{10} \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_0$$

In Matlab, you can input this 10x10 system as

```
A = zeros(10,10);
for i=1:9
   A(i,i) = -20.1;
   A(i,i+1) = 10;
   A(i+1,i) = 10;
   end
A(10,10) = -10.1;
```

```
  -20.1000   10.0000         0         0         0         0         0         0         0         0
   10.0000  -20.1000   10.0000         0         0         0         0         0         0         0
         0   10.0000  -20.1000   10.0000         0         0         0         0         0         0
         0         0   10.0000  -20.1000   10.0000         0         0         0         0         0
         0         0         0   10.0000  -20.1000   10.0000         0         0         0         0
         0         0         0         0   10.0000  -20.1000   10.0000         0         0         0
         0         0         0         0         0   10.0000  -20.1000   10.0000         0         0
         0         0         0         0         0         0   10.0000  -20.1000   10.0000         0
         0         0         0         0         0         0         0   10.0000  -20.1000   10.0000
         0         0         0         0         0         0         0         0   10.0000  -10.1000
```

A is a 10x10 matrix

- It has 10 eigenvalues

Eigenvalues tell you how the system behaves.

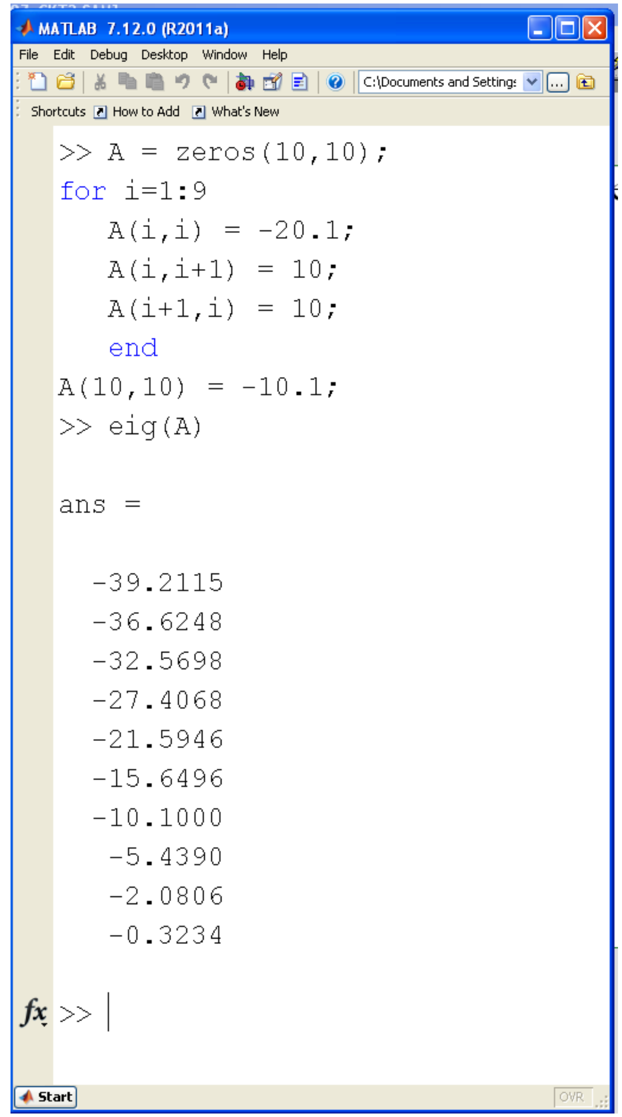
- There is a fast mode which decays as

  $$x(t) = e^{-39.21t}$$

- There is a slow mode which decays as

  $$x(t) = e^{-0.3234t}$$

- There are eight other modes as well



A is a 10x10 matrix

- It also has 10 eigenvectors
- Eigenvectors tell you what behaves that way:

```
>> [a,b] = eig(A);
>> a

a =

   -0.1286   -0.2459    0.3412    0.4063    0.4352    0.4255    0.3780    0.2969   -0.1894    0.0650
    0.2459    0.4063   -0.4255   -0.2969   -0.0650    0.1894    0.3780    0.4352   -0.3412    0.1286
   -0.3412   -0.4255    0.1894   -0.1894   -0.4255   -0.3412   -0.0000    0.3412   -0.4255    0.1894
    0.4063    0.2969    0.1894    0.4352    0.1286   -0.3412   -0.3780    0.0650   -0.4255    0.2459
   -0.4352   -0.0650   -0.4255   -0.1286    0.4063    0.1894   -0.3780   -0.2459   -0.3412    0.2969
    0.4255   -0.1894    0.3412   -0.3412   -0.1894    0.4255    0.0000   -0.4255   -0.1894    0.3412
   -0.3780    0.3780    0.0000    0.3780   -0.3780   -0.0000    0.3780   -0.3780   -0.0000    0.3780
    0.2969   -0.4352   -0.3412    0.0650    0.2459   -0.4255    0.3780   -0.1286    0.1894    0.4063
   -0.1894    0.3412    0.4255   -0.4255    0.3412   -0.1894    0.0000    0.1894    0.3412    0.4255
    0.0650   -0.1286   -0.1894    0.2459   -0.2969    0.3412   -0.3780    0.4063    0.4255    0.4352

>> eig(A)'

  -39.2115  -36.6248  -32.5698  -27.4068  -21.5946  -15.6496  -10.1000   -5.4390   -2.0806   -0.3234
```

Fast Eigenvectror

The first column is the fast mode

- It decays as exp(-39.2115t)

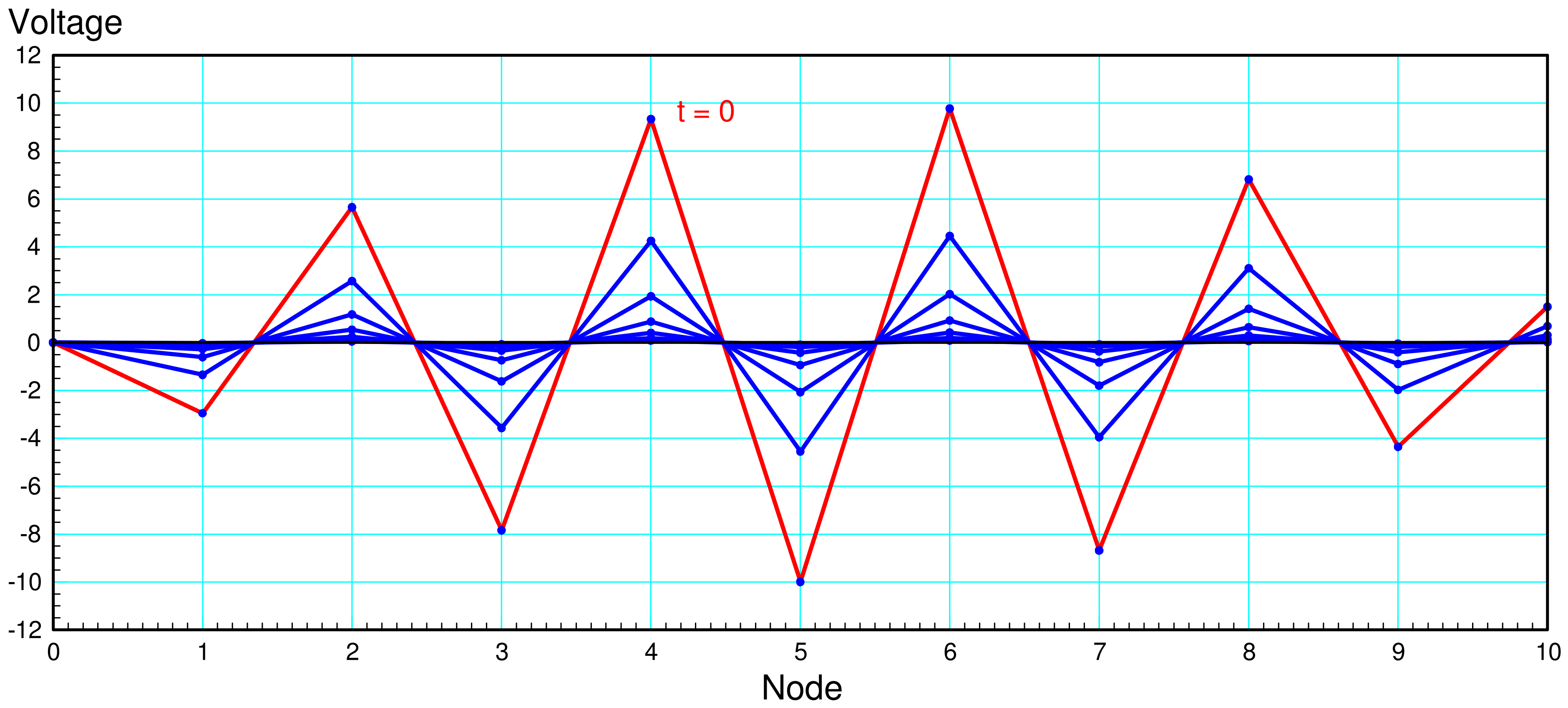
Make the inital condition the fast eigenvector,

- The shape stays the same
- The amplitude drops as exp(-39.2115t)

```
% 10-stage RC Filter

V = [       -2.9558
             5.6490
            -7.8402
             9.3348
           -10.0000
             9.7766
            -8.6845
             6.8208
            -4.3510
             1.4946    ];

dV = zeros(10,1);
V0 = 0;
dt = 0.001;
t = 0;

while(t < 1)

   dV(1)  = 10*V0    - 20.1*V(1) -
   dV(2)  = 10*V(1) - 20.1*V(2) -
   dV(3)  = 10*V(2) - 20.1*V(3) -
```

Natural Response for the Fast Eigenvector

- V0 = 0
- Initial conidtion = fast eigenvector



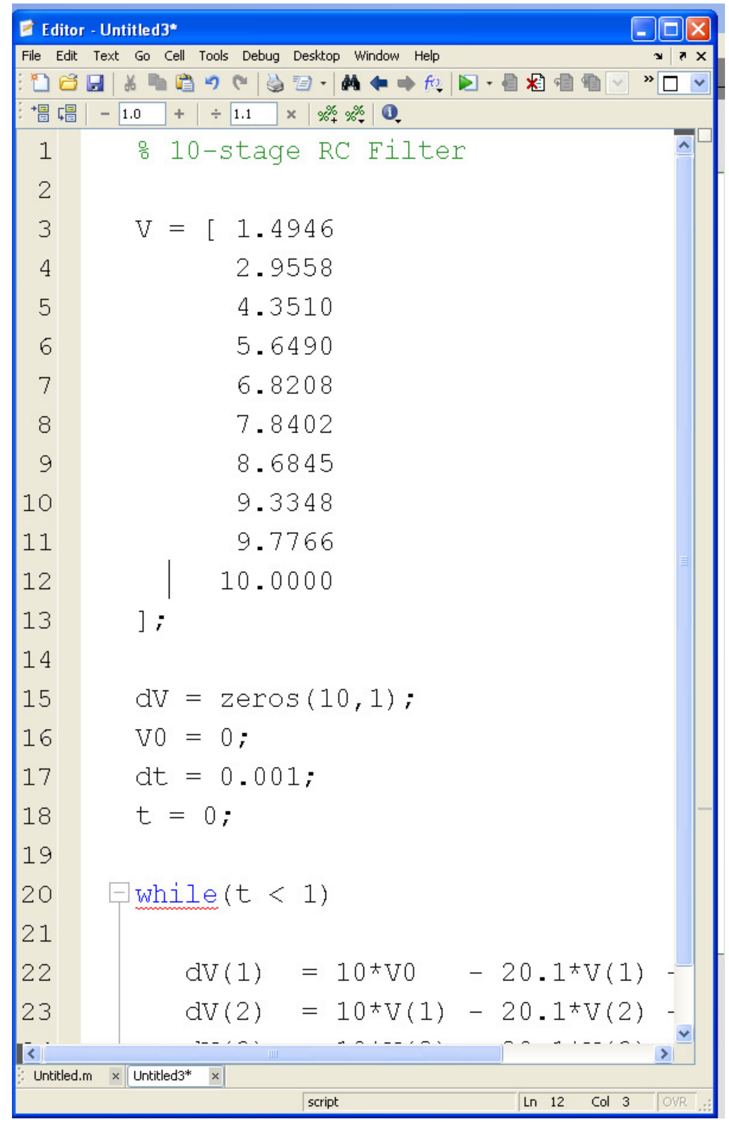
Slow Eigenvector

The last column is the slow eigenvector

- It decays as exp(-0.3234t)

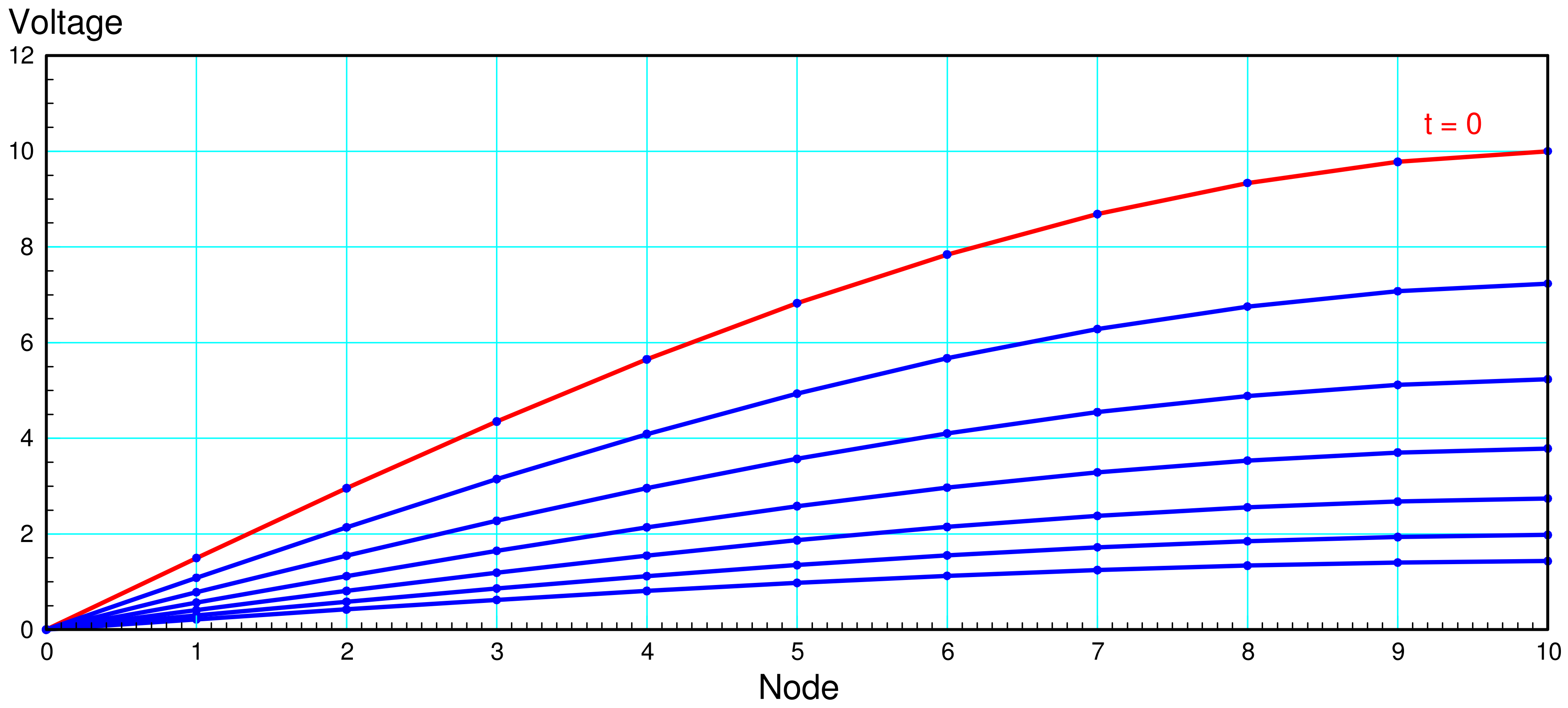
Make the inital condition the slow eigenvector,

- The shape stays the same
- The amplitude drops as exp(-0.3234t)



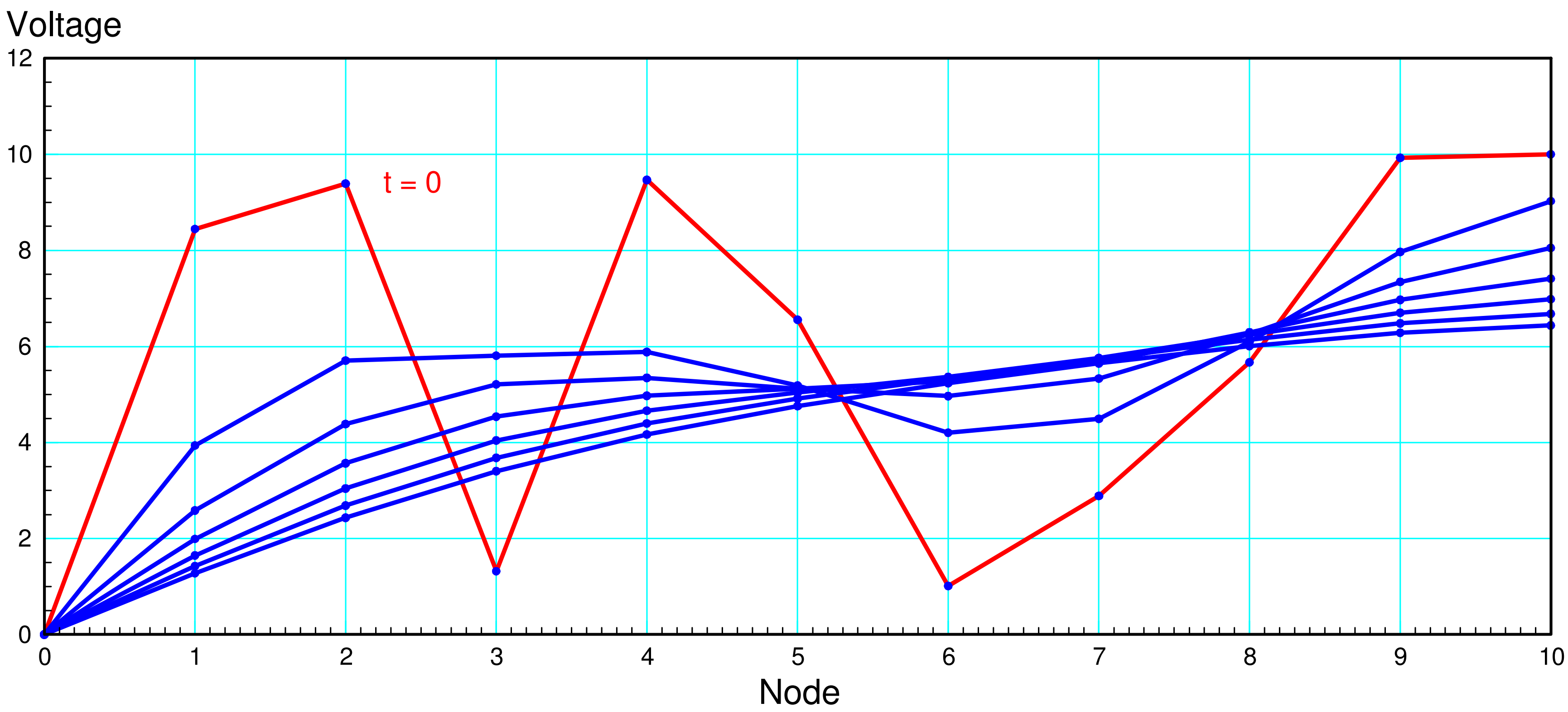
Natural Response for the slow eigenvector

- V0 = 0
- Initial conidtion = slow eigenvector



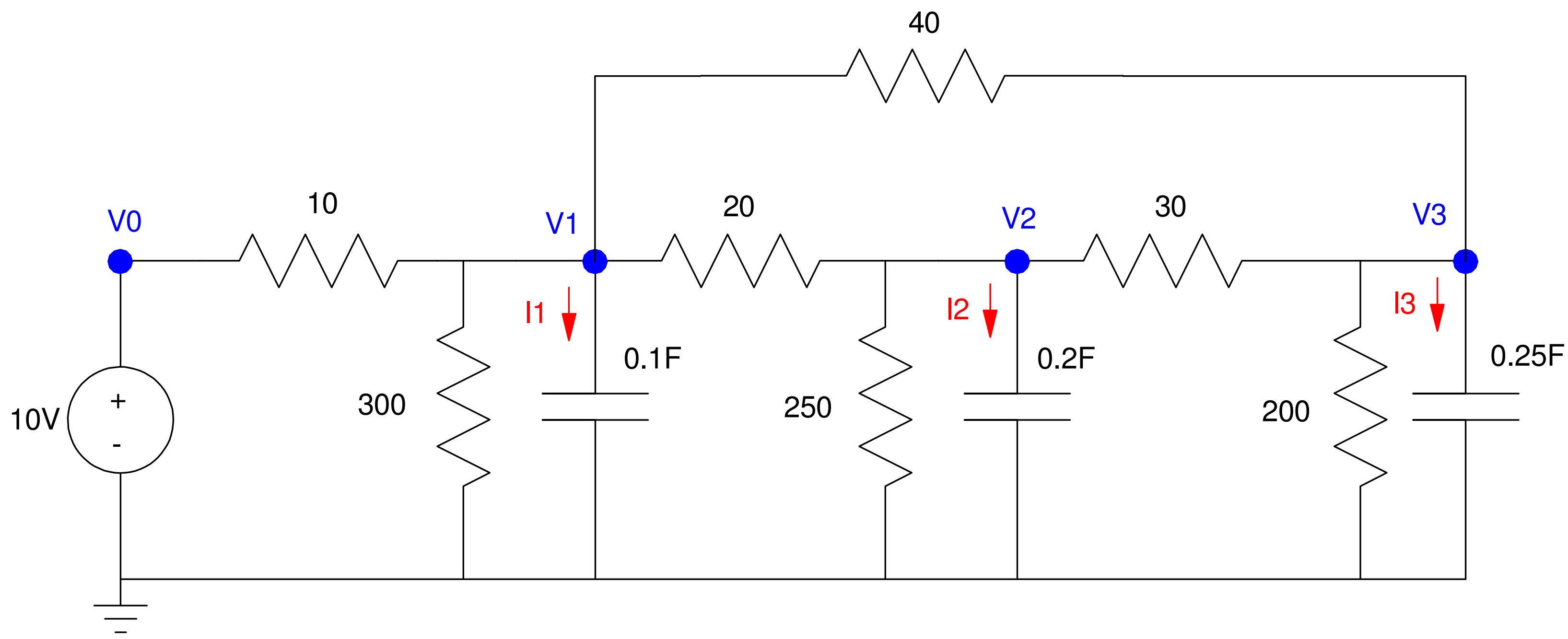
Response for a random initial conidtion

- All 10 eigenvectors are excited
- The fast 9 modes quickly decay
- Leaving the slow (dominant) eigenvector



Practice Problem:

Write the differential equations which describe the following circuit



Summary

Capacitors are integrators

$$V = \frac{1}{C}\int I \cdot dt$$

Differential equations are needed to describe RC circuits

- N capacitors means you need an Nth-order differential equation

Once these differential equations are found, the voltages can be determined using numerical intgration

Eigenvalues tell you *how* the system behaves

Eigenvectors tell you *what* behaves that way