

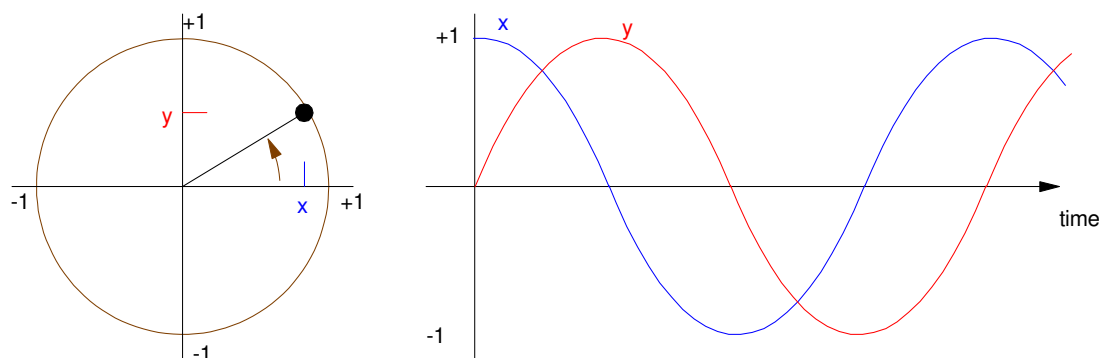
# Sinusoidal Sources

So far, we have looked at constant sources. If you want to look at time-varying signals, such as music, you need to expand our analysis to sinusoidal sources.

## What Is a Sine Wave?

**Circles and Sine Waves:** If you take a wheel and spin it counter clockwise,

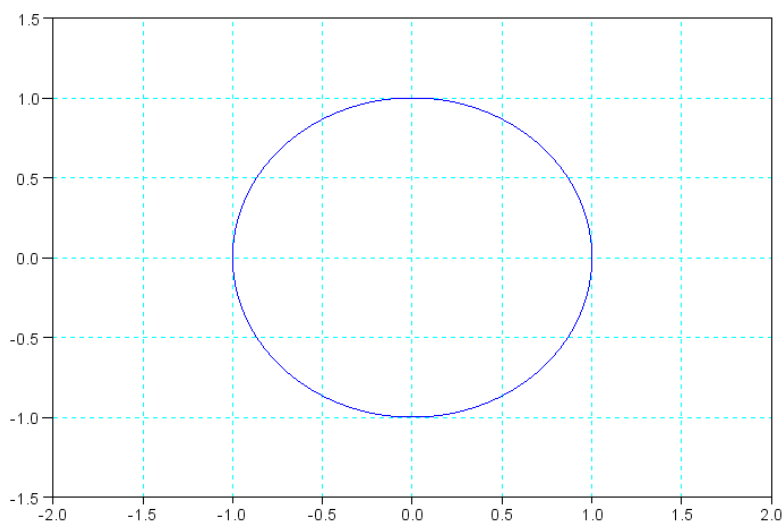
- The x-position of a point on the wheel maps out a cosine function, and
- The y-position of a point maps out a sine function



plotting  $x$  vs.  $y$  where  $x = \cos(q)$   $y = \sin(q)$  produces the unit circle

If you plot  $\cos(q)$  vs.  $\sin(q)$ , you get the unit circle

```
q = [0:0.001:1]' * 2 * pi;  
x = cos(q);  
y = sin(q);  
plot(cos(q), sin(q))
```



plotting  $\cos(q)$  vs.  $\sin(q)$  produces the unit circle

In polar coordinates, if you plot

$$r = \cos \theta$$

or

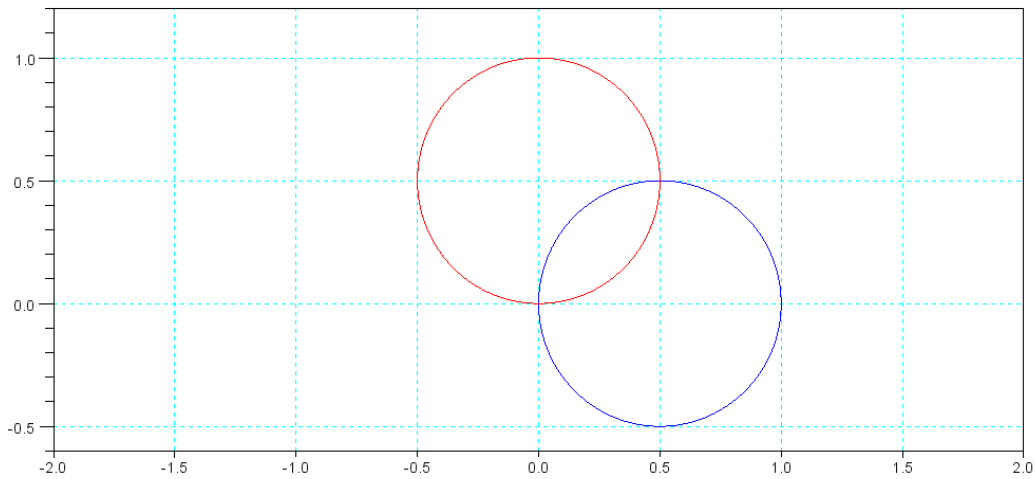
$$r = \sin \theta$$

you also get circles

```
q = [0:0.001:1]' * 2 * pi;

r1 = cos(q);
x1 = r1.*cos(q);
y1 = r1.*sin(q);

r2 = sin(q);
x2 = r2.*cos(q);
y2 = r2.*sin(q);
plot(x1,y1,'b',x2,y2,'r');
```



$r = \cos(q)$  (blue) and  $r = \sin(q)$  (red) also produce circles

**Natural Response to Differential Equations:** The natural response to a linear 2nd-order differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

is a sine wave:

$$y(t) = a \cos(\omega t) + b \sin(\omega t)$$

Example: A mass-spring system satisfies the differential equation:

$$F = m \frac{d^2 y}{dt^2} = -ky$$

Find the response,  $y(t)$  assuming

- $m = 1\text{ kg}$
- $k = 9\text{ N/m}$
- $y(0) = 2\text{ m}$
- $y'(0) = 1\text{ m/s}$

Solution: The differential equation is

$$\frac{d^2 y}{dt^2} = -9y$$

Using methods from Calculus, 'guess' the answer is of the form

$$y(t) = a \cos(\omega t) + b \sin(\omega t)$$

Take the 1st derivative:

$$\frac{dy}{dt} = -a\omega \sin(\omega t) + b\omega \cos(\omega t)$$

Take the 2nd derivative

$$\begin{aligned} \frac{d^2 y}{dt^2} &= -a\omega^2 \cos(\omega t) - b\omega^2 \sin(\omega t) \\ &= -\omega^2 (a \cos(\omega t) + b \sin(\omega t)) \\ &= -\omega^2 y \end{aligned}$$

Substitute

$$\begin{aligned} \frac{d^2 y}{dt^2} &= -9y \\ -\omega^2 y &= -9y \\ \omega &= 3 \end{aligned}$$

and

$$y(t) = a \cos(3t) + b \sin(3t)$$

Plugging in the initial conditions:

$$\begin{aligned} y(0) &= 2 = a \\ y'(0) &= 1 = -3b \end{aligned}$$

This results in

$$y(t) = 2 \cos(3t) - 0.333 \sin(3t)$$

## Why Use Sine Waves?

There are two main reasons we use sine waves (instead of square waves or other time-varying signals)

- Sine waves are eigenfunctions: if you have a differential equation whose input is a sine wave, the output is a sine wave. Not many functions have that property.
- You can decompose any periodic signal into a sum of sine waves (Fourier transform - coming soon). If you can solve a circuit for a sinusoidal input, you can solve it for any periodic input.

**Eigenfunctions:** Sine waves are the only functions where the solution to a differential equation is the same form as that function. For example, take the differential equation

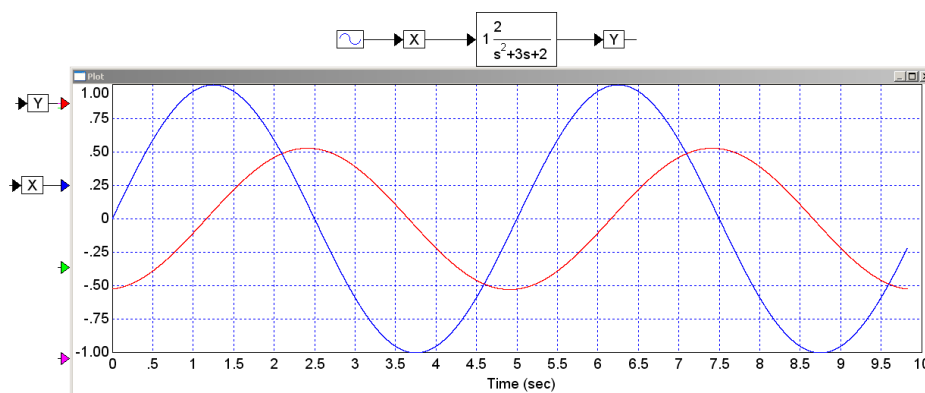
$$y'' + 3y' + 2y = 2x$$

In transfer function form, this can be written as

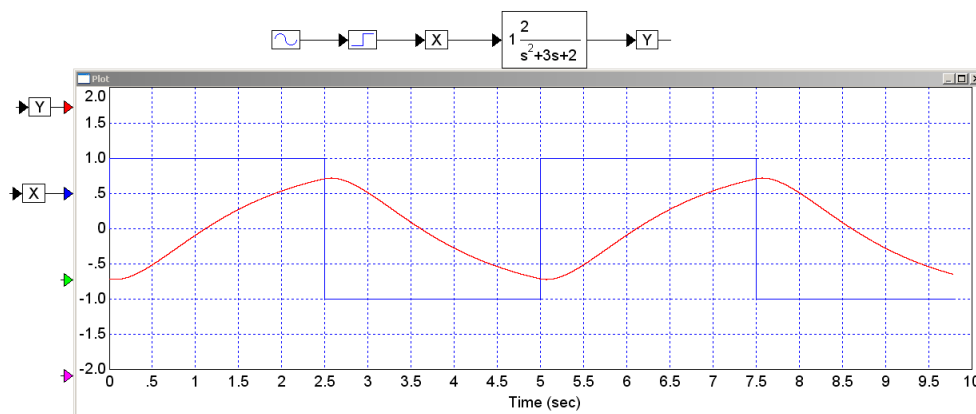
$$(s^2 + 3s + 2)Y = 2X$$

$$Y = \left( \frac{2}{s^2 + 3s + 2} \right) X$$

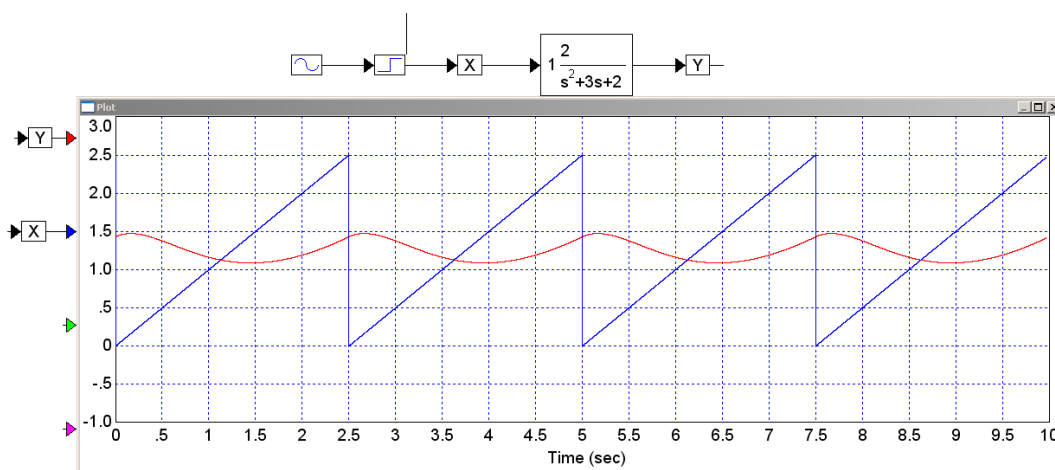
Later, we'll see that this describes a 2-stage RC filter (as well as other systems). If the input,  $X$ , is a pure sine wave, the output is a pure sine wave at the same frequency:



No other periodic function has this property: square waves don't do this:



Sawtooth waves don't do this:



Only sine waves have the property that the output is the same shape as the input, only with a change in amplitude and with a time delay.

### Fourier Transform: (Covered in ECE 311 Circuits II)

A second reason for using sine waves is that they are versatile: you can represent any periodic signal as a sum of sine waves.

For example, a half-rectified sine wave

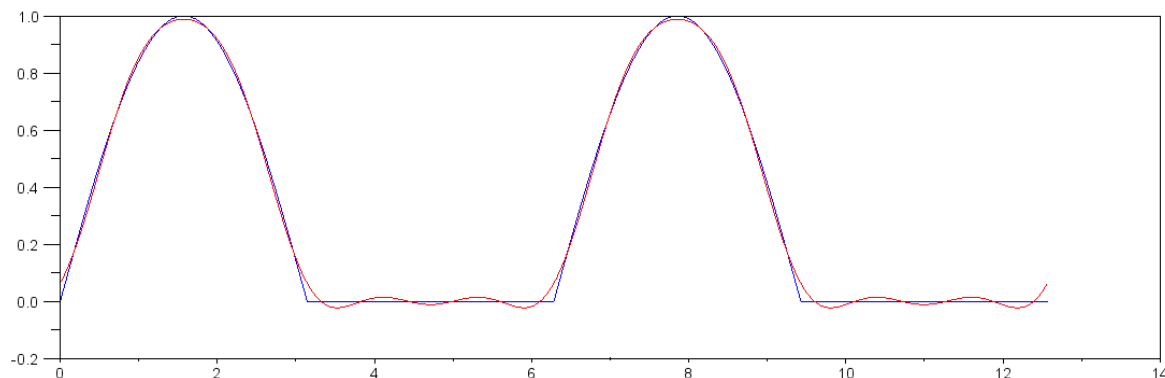
$$x(t) = \begin{cases} \sin(t) & \sin(t) > 0 \\ 0 & \text{otherwise} \end{cases}$$

can be expressed as the series

$$x(t) = \frac{1}{\pi} + \frac{1}{2} \sin(t) + \sum_{n \text{ even}} \left( \frac{2}{\pi(n^2-1)} \right) \cos(nt)$$

```
t = [0:0.001:2]' * 2*pi;
x = max(0, sin(t));

y = 1/pi + 0.5*sin(t) - 2/(3*pi)*cos(2*t) - 2/(15*pi)*cos(4*t);
plot(t,x,'b',t,y,'r')
```



## Sine Wave Definitions

To alleviate some of the confusion, some definitions are needed.

**V<sub>p</sub>: Peak Voltage:** The amplitude of the sine wave from it's average voltage (usually zero).

**V<sub>pp</sub>: Peak to Peak Voltage:** The distance between the maximum and minimum voltage.  $V_{pp} = 2 V_p$

**V<sub>rms</sub>: rms Voltage:** The DC voltage which would produce the same amount of heat through a 1 Ohm resistor.

**Period (seconds):** Time time between zero crossings (or peak voltages)

**Frequency (Hz):** One over the period

**Frequency (rad/sec):** The natural frequency:  $1\text{Hz} = 2\pi \text{ rad/sec}$ .

You can convert from one to the other

$$V_p = \frac{1}{2} V_{pp}$$

$$V_{rms} = \frac{1}{\sqrt{2}} V_p$$

Derivation of rms voltage: Assume  $V$  is 1 rad/sec a sine wave

$$v(t) = V_p \sin(t)$$

The instantaneous power through a 1 Ohm resistor is

$$P(t) = \frac{v^2}{R} = (V_p \sin(t))^2$$

The average power through a 1 Ohm resistor is

$$P_{avg} = \frac{1}{2\pi} \int_0^{2\pi} (V_p \sin(t))^2 dt$$

$$P_{avg} = \frac{1}{2\pi} \cdot \int_0^{2\pi} \left( V_p^2 \cdot \left( \frac{1 - \cos(2t)}{2} \right) \right) dt$$

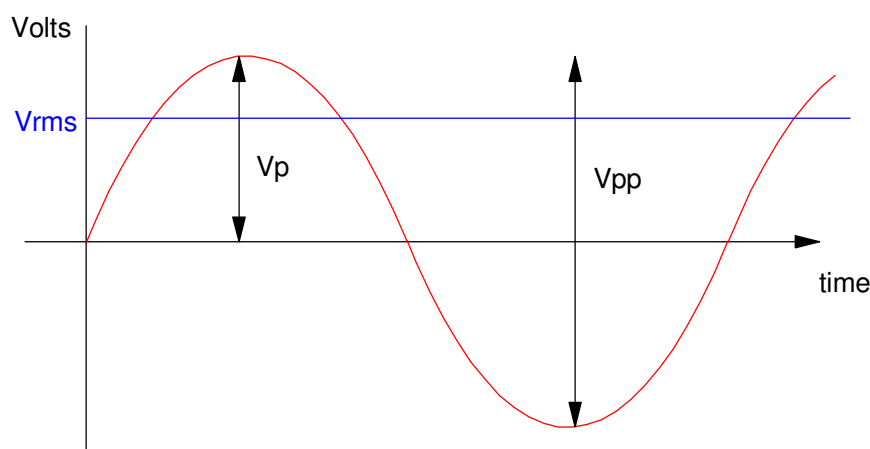
$$P_{avg} = \frac{V_p^2}{2}$$

The DC voltage which produces the same power is the rms voltage

$$\left( \frac{V_{rms}^2}{R} \right) = V_{rms}^2 = \frac{1}{2} V_p^2$$

$$V_{rms} = \frac{1}{\sqrt{2}} V_p$$

$V_p$ ,  $V_{pp}$ , and  $V_{rms}$  are different. Likewise, when dealing with sinusoids, you have to specify what units you're talking about.



Relationship between  $V_p$ ,  $V_{pp}$ , and  $V_{rms}$

The same holds for current:

**$I_p$ : Peak Current** The maximum current relative to the average current

**$I_{pp}$ : Peak-to-Peak Current:** The maximum current minus the minimum current

**$I_{rms}$ : rms Current:** The DC current which produces the same amount of heat through a 1 Ohm resistor.