
Capacitors

ECE 211 Circuits I

Lecture #18

Please visit [Bison Academy](#) for corresponding lecture notes, homework sets, and solutions

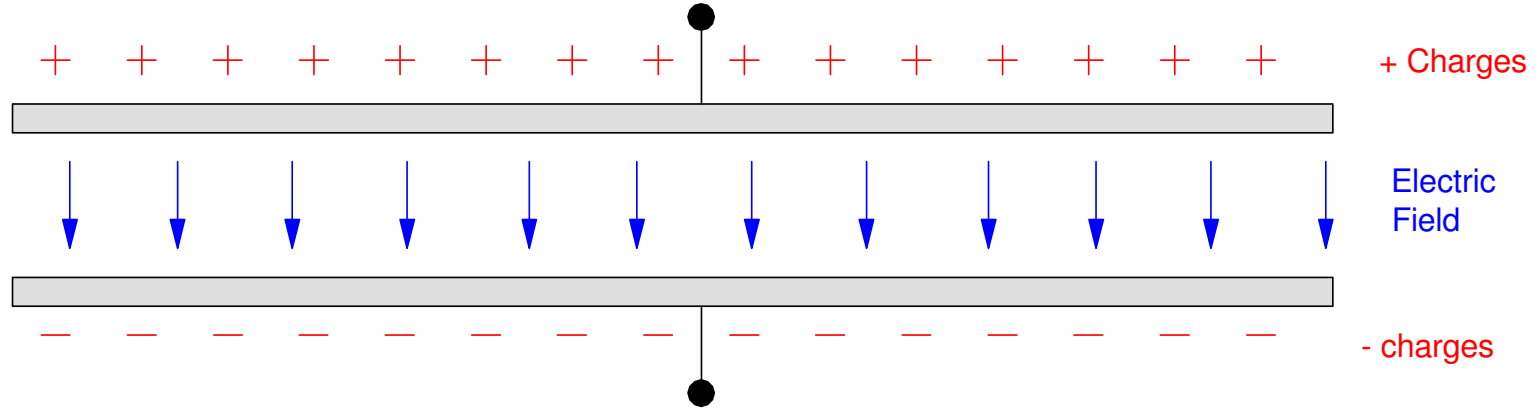
Capacitors

A capacitor is a set of parallel plates¹ with the capacitance equal to

$$C = \epsilon \frac{A}{d} \text{ (Farads)}$$

where

- ϵ is the dielectric constant of the material between plates (air = $8.84 \cdot 10^{-12}$)
- A is the area of the capacitor, and
- d is the distance between plates.



¹ <http://www.electronics-tutorials.ws/>

1 Farad

The area you need for 1 Farad with plates 1mm apart is

$$1 = (8.84 \cdot 10^{-12}) \frac{A}{0.001m}$$

$$A = 113,122,171m^2$$

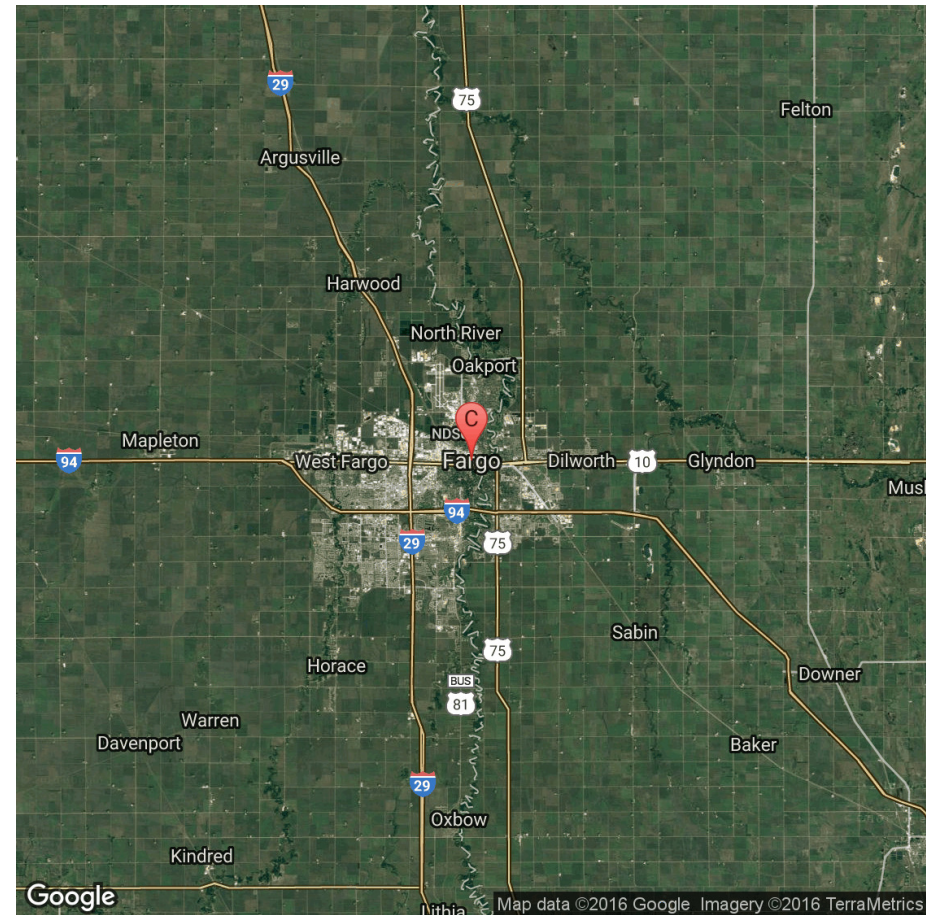
Equal to 10.6km x 10.6km

- About the size of Fargo
- Most capacitors on the order of μF or nF

The charge stored is

$$Q = C \cdot V$$

Q = charge (Coulombs - one Coulomb is equal to $6.242 \cdot 10^{18}$ electrons).



Voltage - Current Relationship

$$Q = CV$$

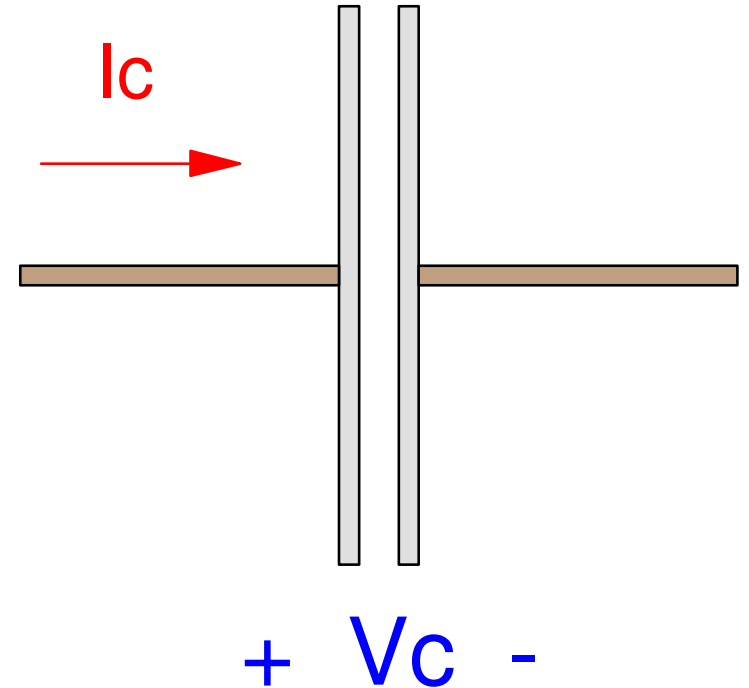
$$I = \frac{dQ}{dt} = C \frac{dV}{dt} + V \frac{dC}{dt}$$

Assuming the capacitance is constant

$$I = C \frac{dV}{dt}$$

This means that capacitors are integrators:

$$V = \frac{1}{C} \int I \cdot dt$$



Capacitors and Energy Storage

The energy stored is

- $E = \frac{1}{2}CV^2$
- 12.5J = 12.5kW for 1ms
- Capacitors provide energy for short bursts

Item	Energy (Joules)	Cost	\$ / MJ
1 pound Wyoming Coal	3,600,000	\$0.028	\$0.0078
1 pound ND Lignite	1,565,217	\$0.017	\$0.0108
1 pint of gasoline	15,000,000	\$0.37	\$0.0247
Lithium battery (D cell)	246,240	\$6	\$24.4
1F Capacitor (5V)	12.5	\$1.53	\$122,400

Numerical Integration and Capacitors

Capacitors are inherently integrators:

$$I = C \frac{dV}{dt}$$

$$V = \frac{1}{C} \int I dt$$

Differential equations are required to described circuits with capacitors

- Each capacitor adds a 1st-order differential equation
- N capacitors means an Nth-order differential equation

Calculus:

- Solving differtial equations using calculus

Circuits I:

- Solving differential equations using numerical methods
- Solving differetial equations using phasors (coming soon)

Circuits II, Signals & Systems:

- Solving differential equtions using LaPlace transforms
-

Numerical Integration

- Solve a differential equation using numerical methods (i.e. Matlab)
- Whole field of mathematics deals with numerical integration

Several types of numerical integration

- Euler Integration
- Trapezoid Rule
- Runge Kutta Integration
- more...

All are approximate

- Use Calculus or LaPlace transforms to get closed-form, exact solutions

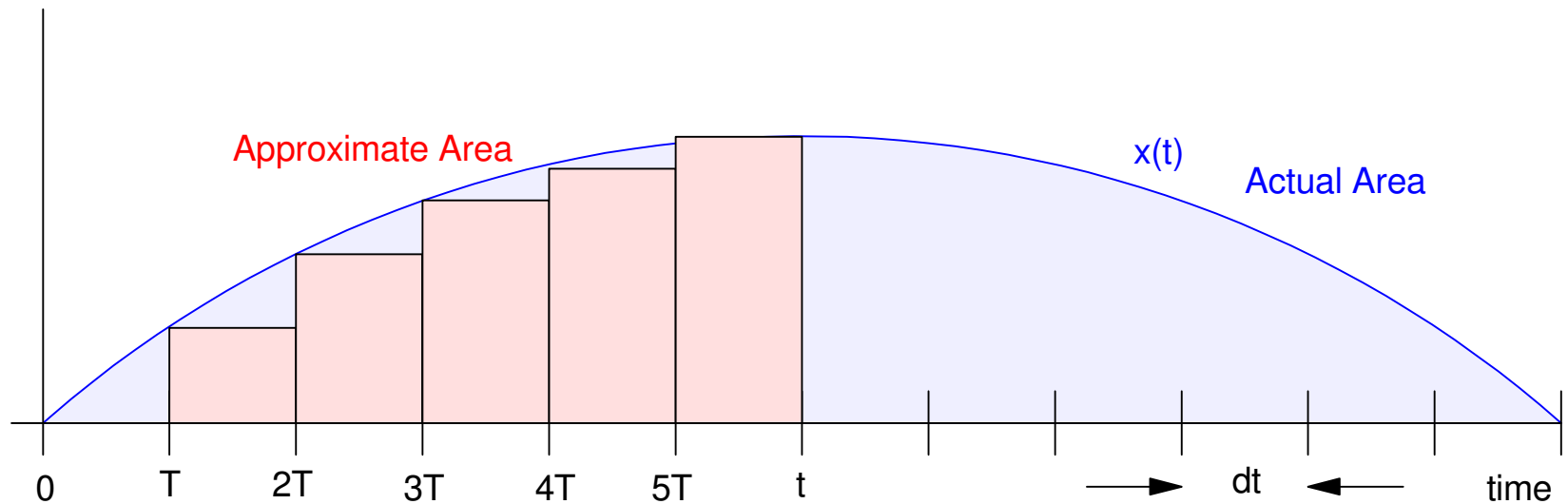


Euler Integration

- $y(t) = y(t - T) + x(t) \cdot dt$
- Sample $x(t)$ every T seconds,
- Use rectangles to approximate the area every T seconds, and
- Sum up the area of each rectangle.

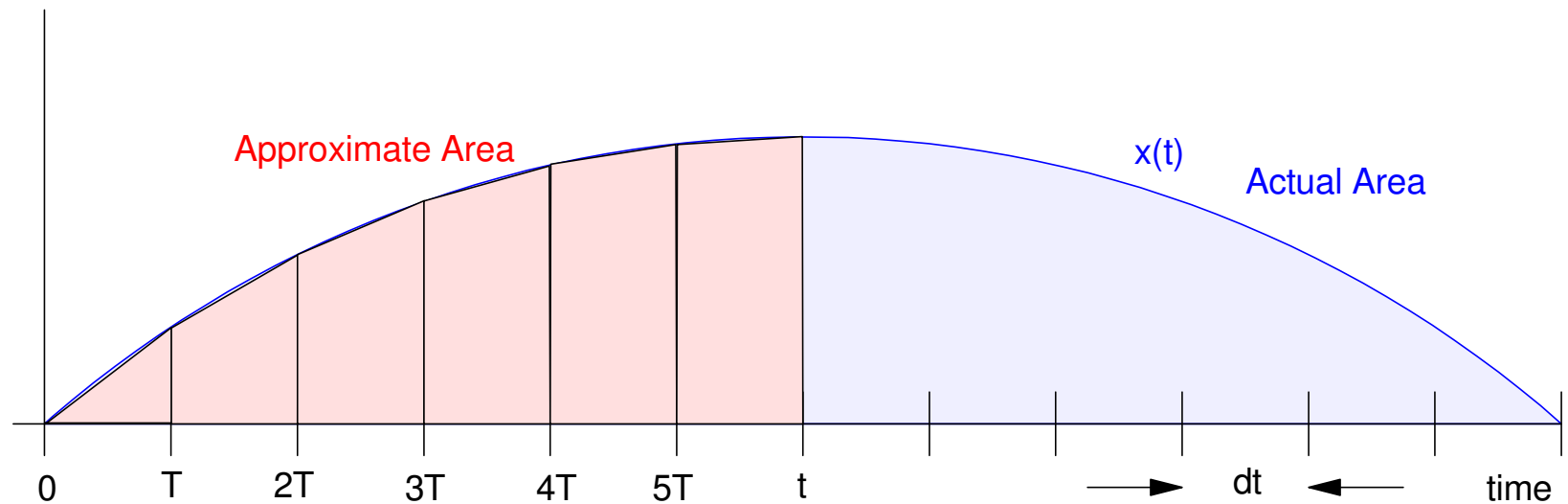
Result is the simplest and least accurate of the three forms.

- Not too bad if you keep the sampling time (dt) small.



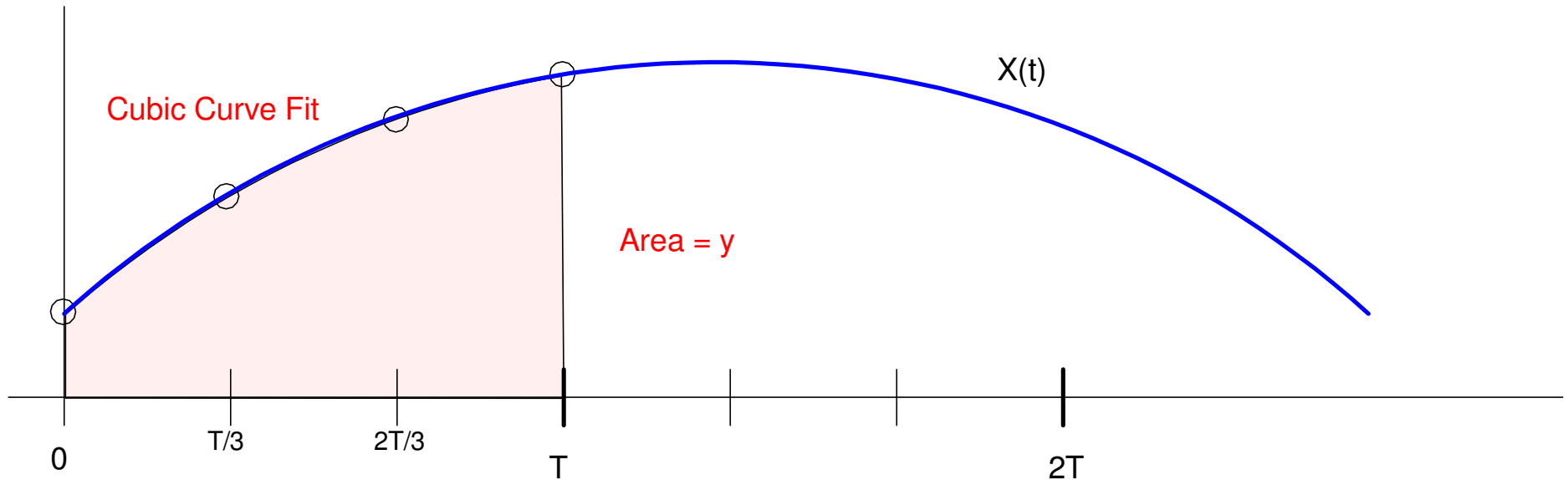
Trapezoid (Bilinear) Integration:

- $y(t) = y(t - T) + \left(\frac{x(t) + x(t-T)}{2} \right) \cdot dt$
- Sample $x(t)$ every T seconds,
- Use trapezoids to approximate the area every T seconds
- *Much* better than Euler
- Required memory (need to recall previous input)



Runge Kutta Integration:

- Sample $x(t)$ every T seconds,
- Use parabola, cubics, etc. to approximate the area
- Required memory and data inbetween samples



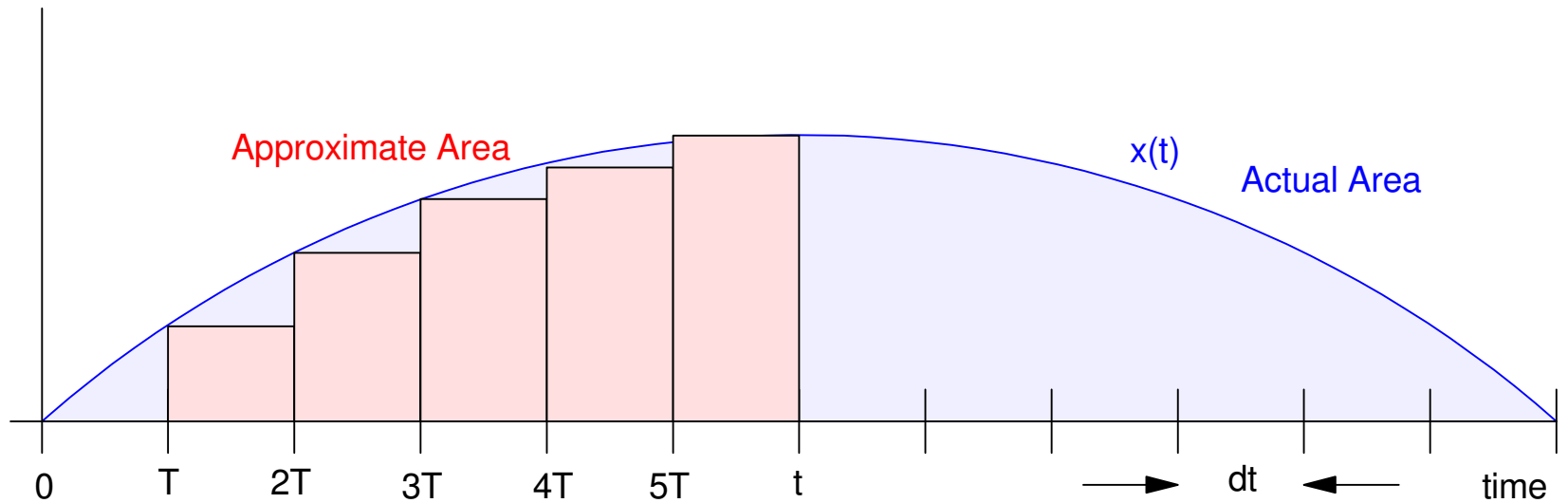
Stick with Euler integration

To find the voltage across a capacitor

- Compute the current to the capacitor, and
- Integrate using Euler integration:

$$dV = I / C$$

$$V = V + dV * dt$$



1-Stage RC Circuit

Find $V_1(t)$ with

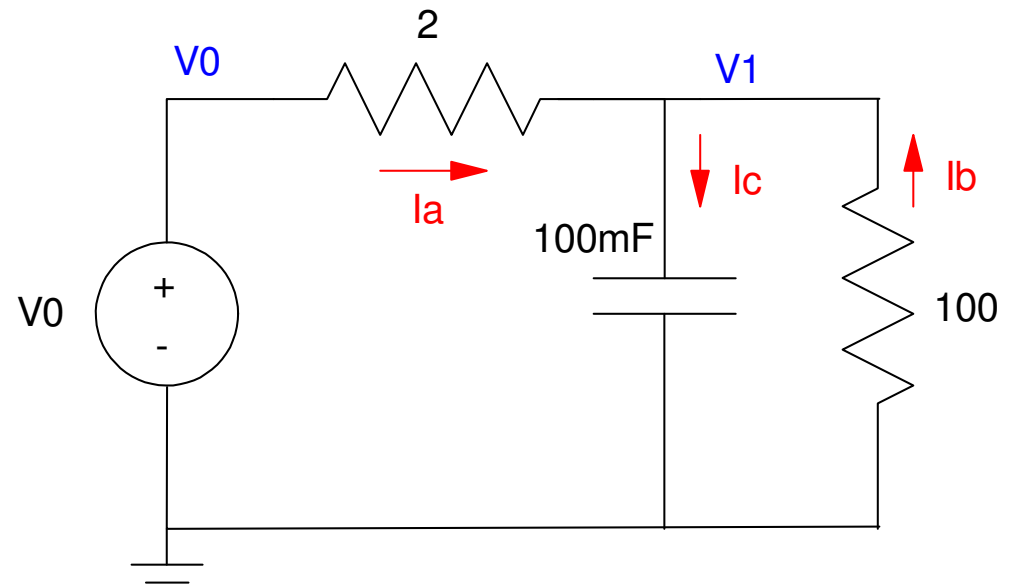
$$V_0(t) = 10u(t) = \begin{cases} 0V & t < 0 \\ 10V & t > 0 \end{cases}$$

Solution

$$I_c = I_a + I_b$$

$$C \frac{dV_1}{dt} = I_c = \left(\frac{V_0 - V_1}{2} \right) + \left(\frac{0 - V_1}{100} \right)$$

$$\frac{dV_1}{dt} = -5.1 V_1 + 5 V_0$$



Solve in Matlab

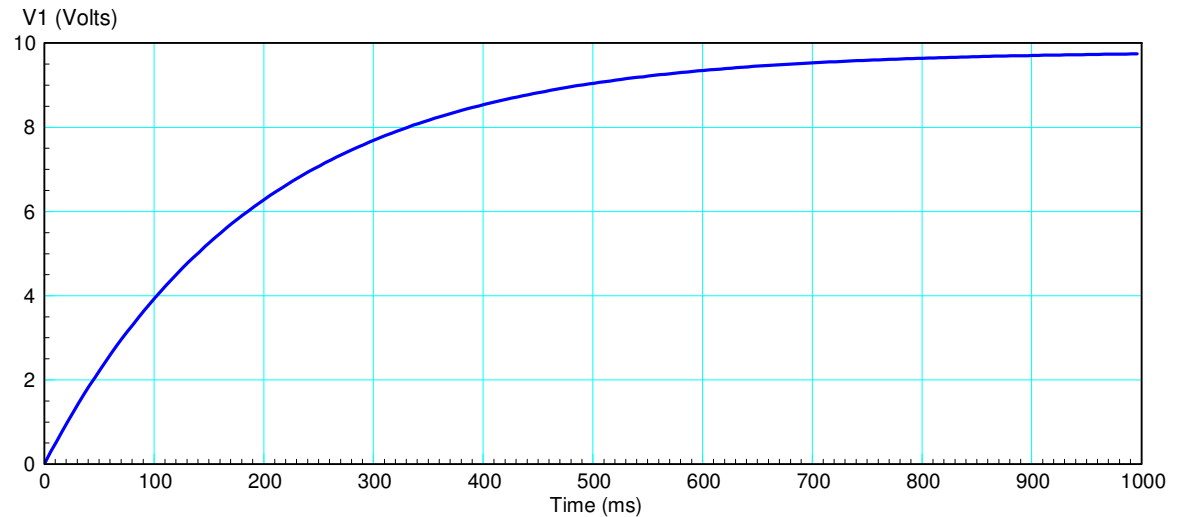
```
% 1-stage RC Filter

V = 0;
V0 = 10;
dt = 0.01;
t = 0;
Y = [];

while(t < 1)
    dV = -5.1*V + 5*V0;
    V = V + dV*dt;
    t = t + dt;
    Y = [Y ; V];
end

t = [1:length(Y)]' * dt;

plot(t, Y);
```



CircuitLab

V0

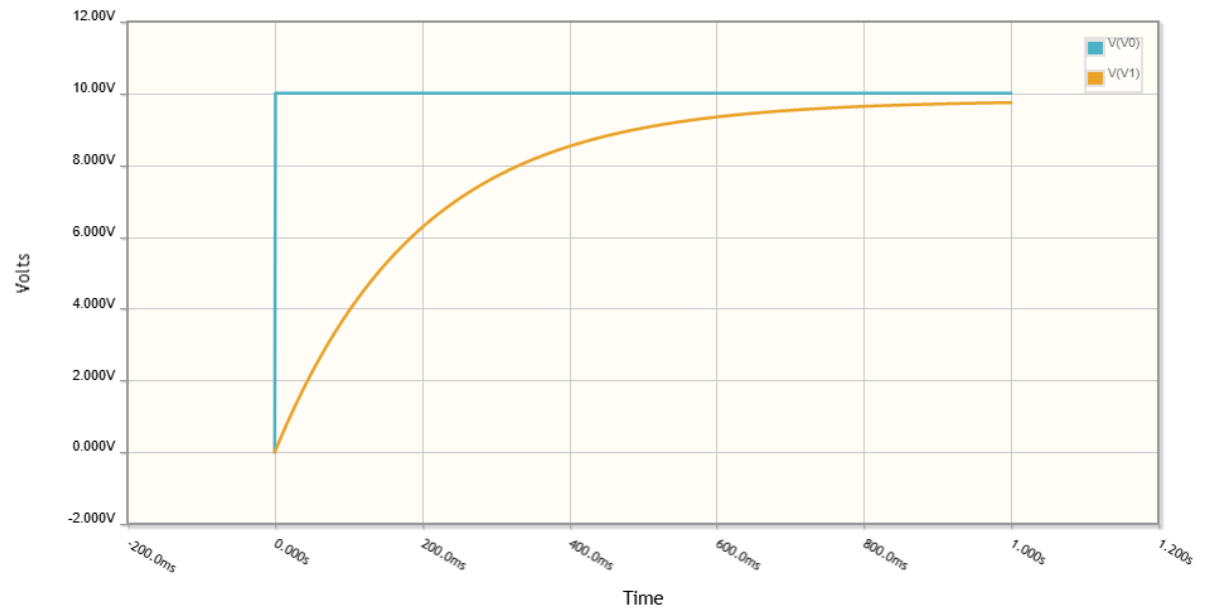
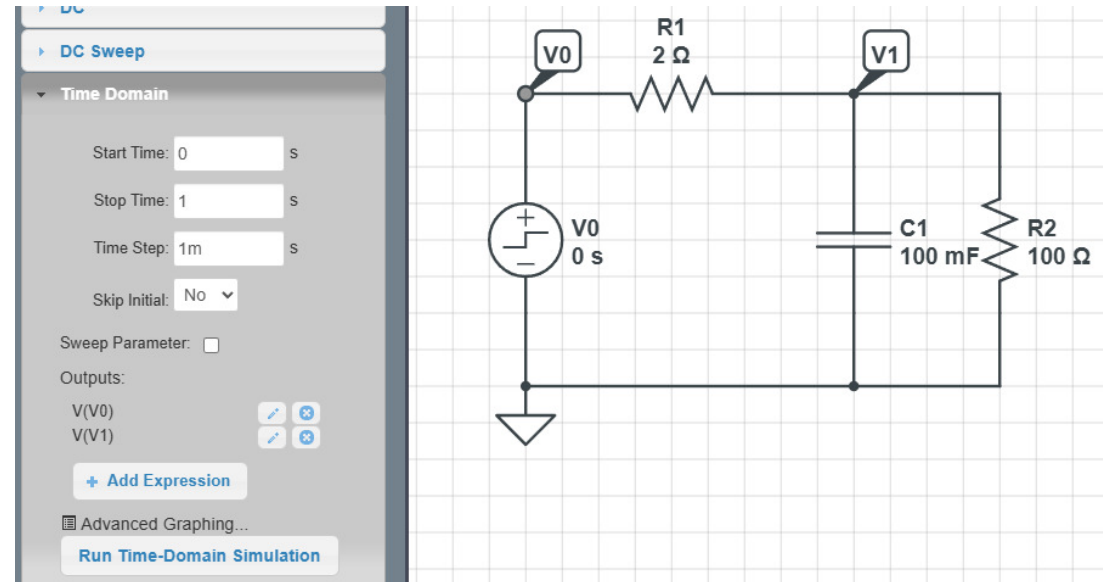
- 10V step input

Simulate

- Stop time: 1 second
- Step Size: 1ms
- Gives 1000 points in the plot

Same result as Matlab

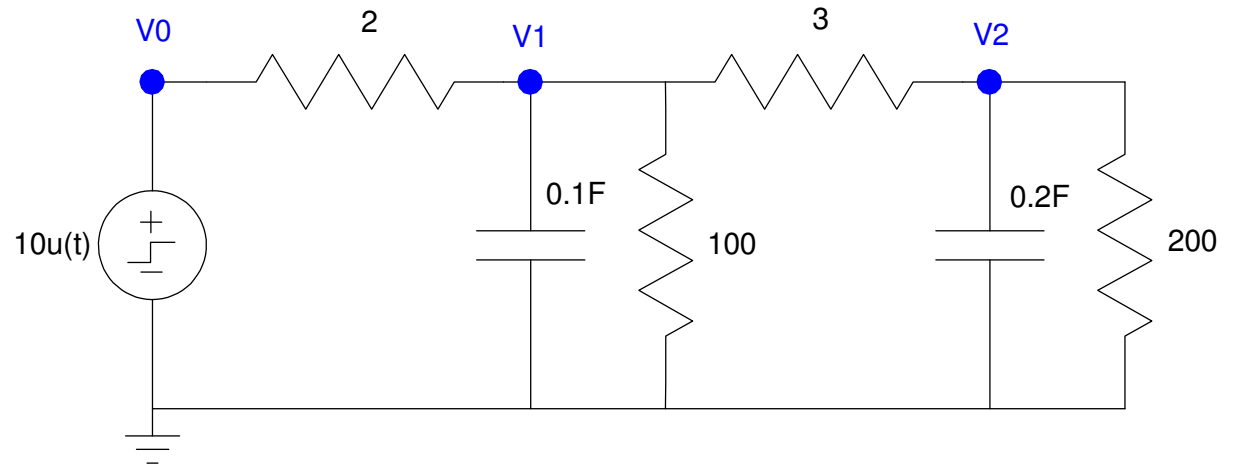
- CircuitLab also solves using numerical integration



2-Stage RC Filter

Write the differential equations

- $V_0 = 0 \quad t < 0$
- $V_0 = 10 \quad t > 0$



2-Stage RC Filter

Differential Equations

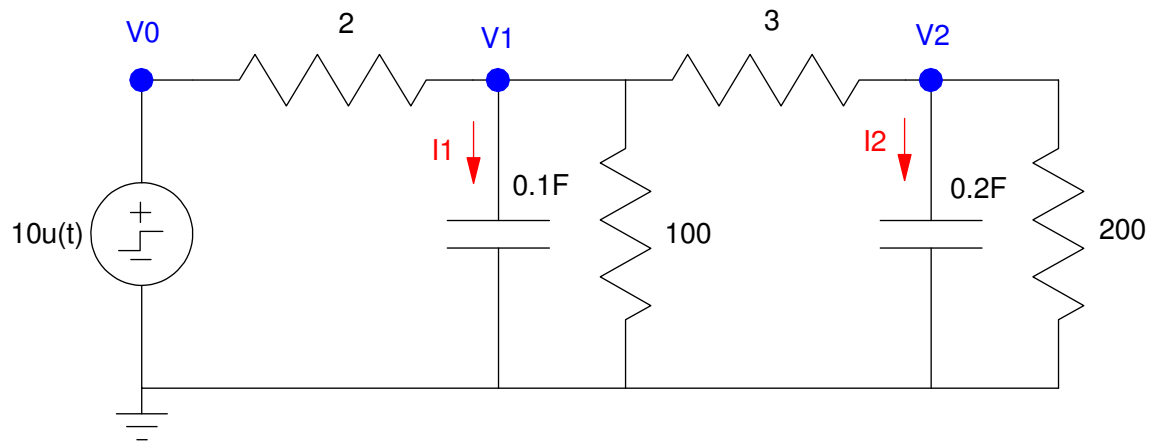
$$I_1 = 0.1 \frac{dV_1}{dt} = \left(\frac{V_0 - V_1}{2} \right) + \left(\frac{V_2 - V_1}{3} \right) - \left(\frac{V_1}{100} \right)$$

$$I_2 = 0.2 \frac{dV_2}{dt} = \left(\frac{V_1 - V_2}{3} \right) - \left(\frac{V_2}{200} \right)$$

Simplify

$$\frac{dV_1}{dt} = 5V_0 - 8.433V_1 + 3.33V_2$$

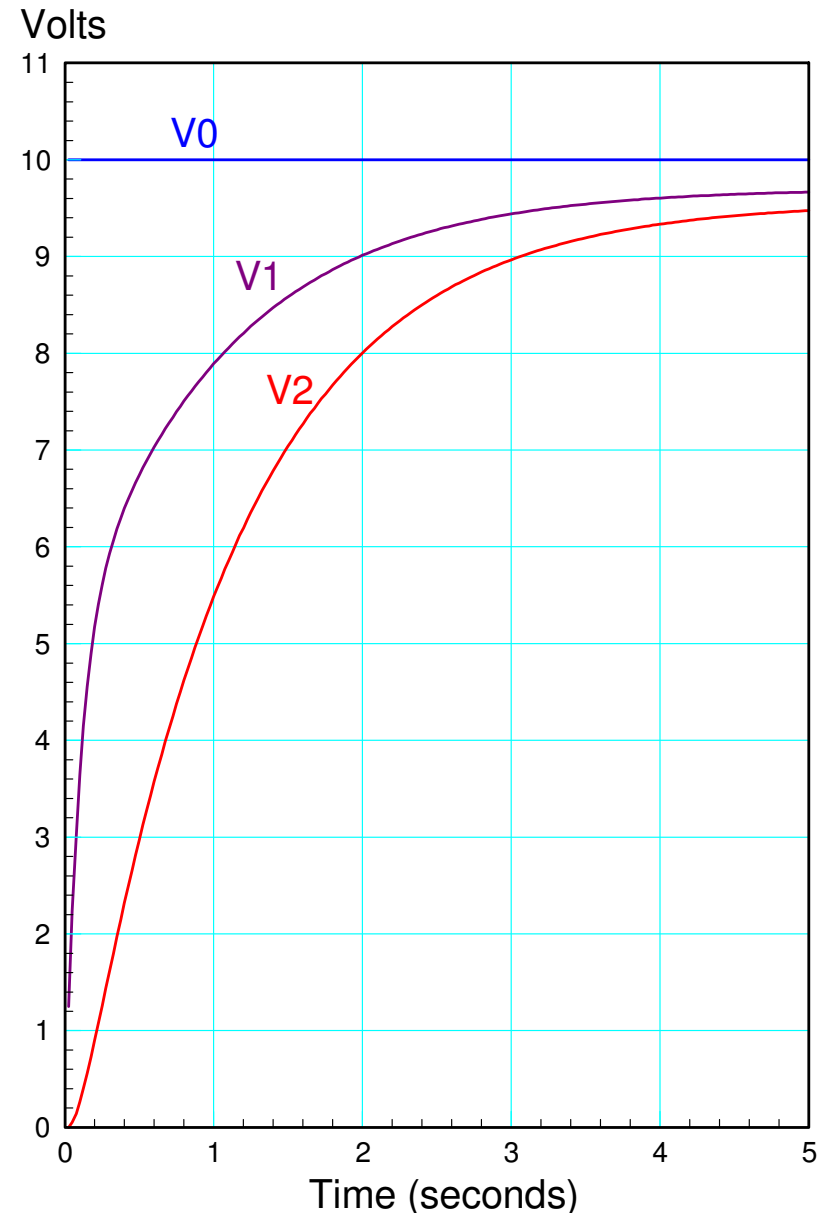
$$\frac{dV_2}{dt} = 1.667V_1 - 1.692V_2$$



2-Stage RC Filter

Matlab Solution

```
V1 = 0;  
V2 = 0;  
V0 = 10;  
t = 0;  
dt = 0.01  
y = [];  
while(t < 5)  
    dV1 = 5*V0 - 8.433*V1 +  
3.33*V2;  
    dV2 = 1.667*V1 - 1.692*V2;  
  
    V1 = V1 +dV1*dt;  
    V2 = V2 + dV2*dt;  
    t = t + dt;  
    y = [y ; t, V0, V1, V2];  
end  
  
plot(y(:,1),y(:,2:4))
```

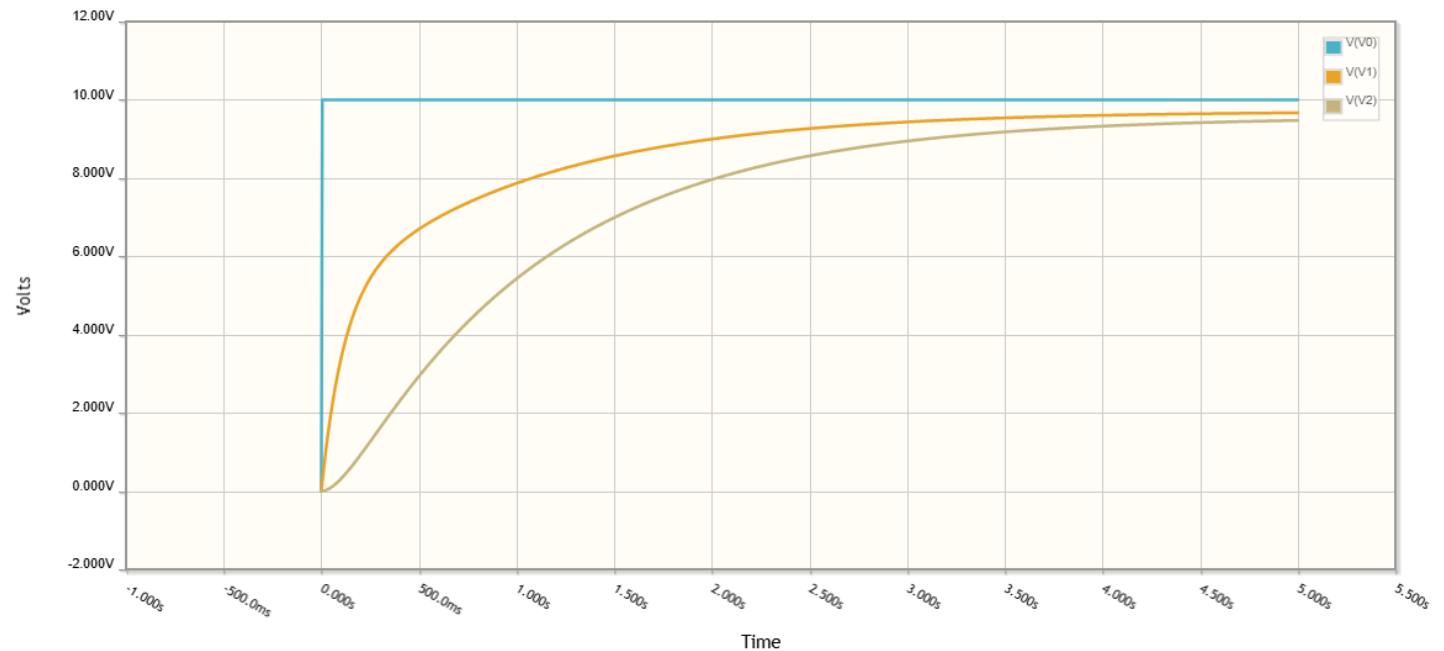
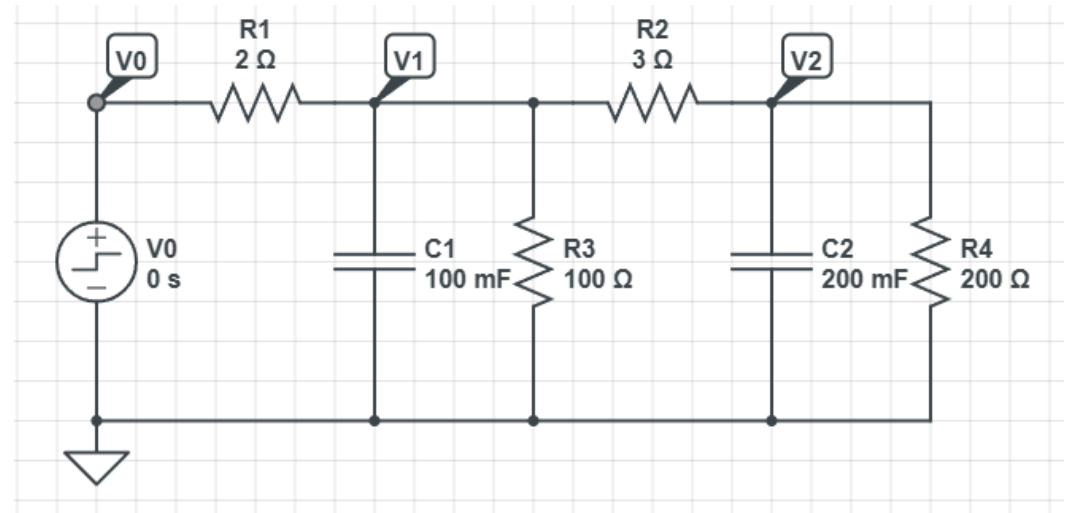


2-Stage Filter

CircuitLab Solution

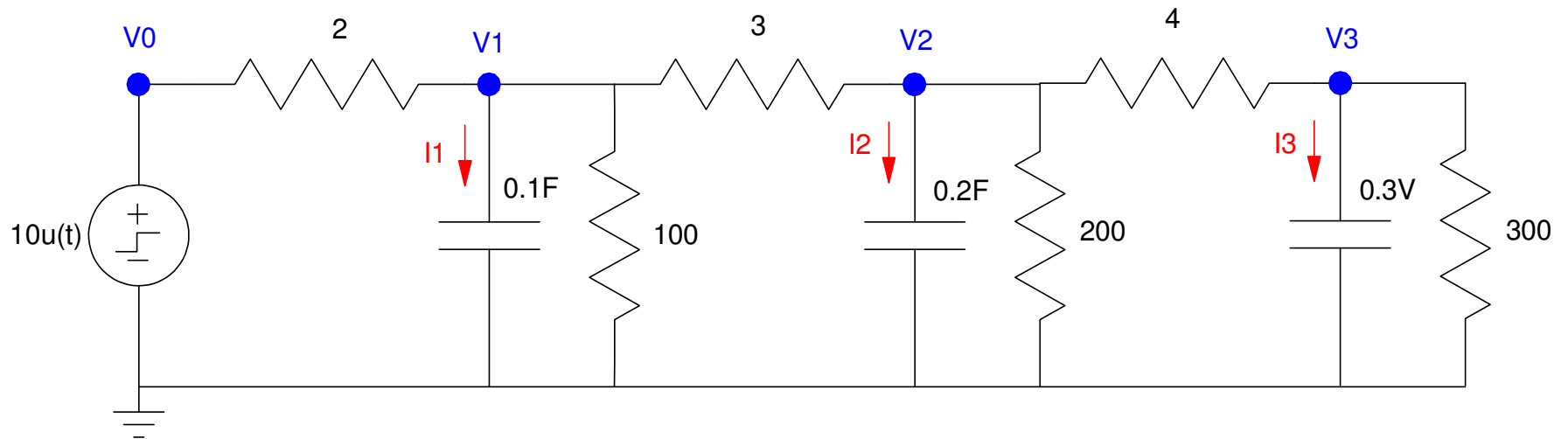
- Same as Matlab

CircuitLab uses numerical integration to find $V(t)$



3-Stage RC Filter

Write the differential equations



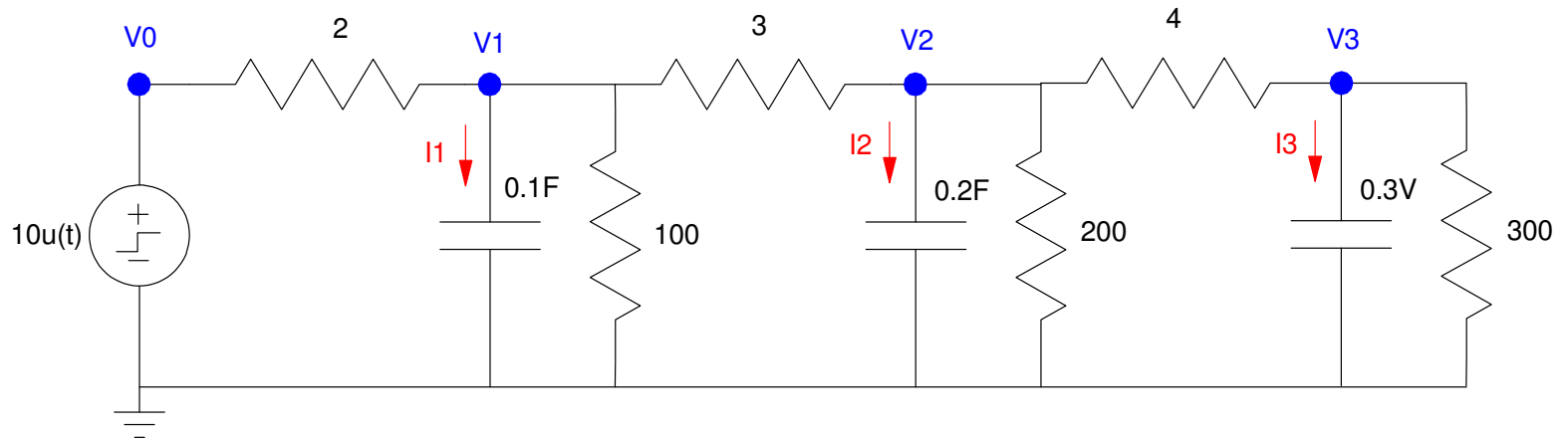
3-Stage RC Filter

$$V_0(t) = 10u(t)$$

$$0.1 \frac{dV_1}{dt} = I_1 = \left(\frac{V_0 - V_1}{2} \right) - \left(\frac{V_1}{100} \right) + \left(\frac{V_2 - V_1}{3} \right)$$

$$0.2 \frac{dV_2}{dt} = I_2 = \left(\frac{V_1 - V_2}{3} \right) + \left(\frac{0 - V_2}{200} \right) + \left(\frac{V_3 - V_2}{4} \right)$$

$$0.3 \frac{dV_3}{dt} = I_3 = \left(\frac{V_2 - V_3}{4} \right) + \left(\frac{0 - V_3}{300} \right)$$



Eigenvalues & Eigenvectors

- New Topic

Go back to the 2-stage RC filter

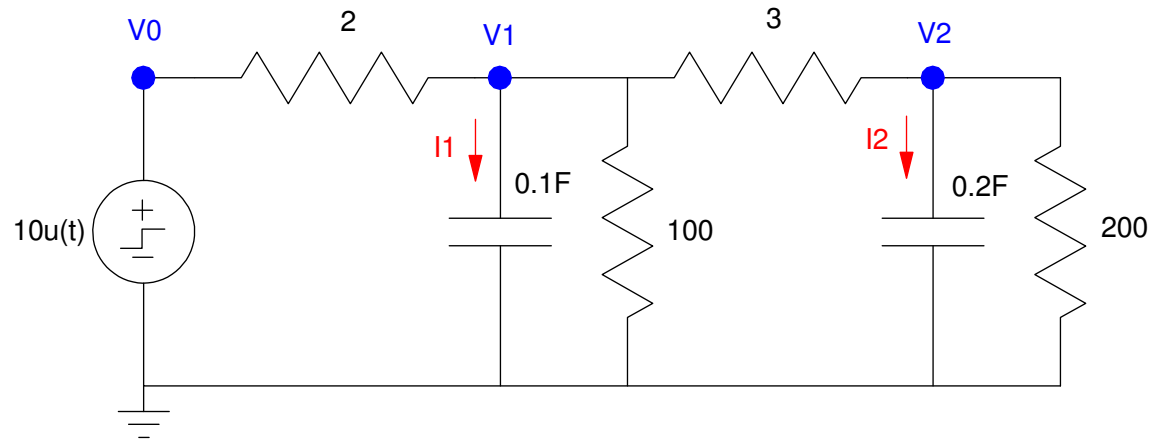
- Assume $V_0 = 0$
- Assign a voltage to V_1 & V_2

What initial conditions

- Decay as quickly as possible?
- Decay as slowly as possible?
- How fast do they decay?

This is an eigenvalue & eigenvector problem

- Throwback to Linear Algebra



Eigenvalues & Eigenvectors

- Math 129 Linear Algebra definitions

Eigenvalues are the solution to

$$|A - \lambda I| = 0$$

Eigenvectors are the solution to

$$A\Lambda = \lambda\Lambda$$

What's this mean?

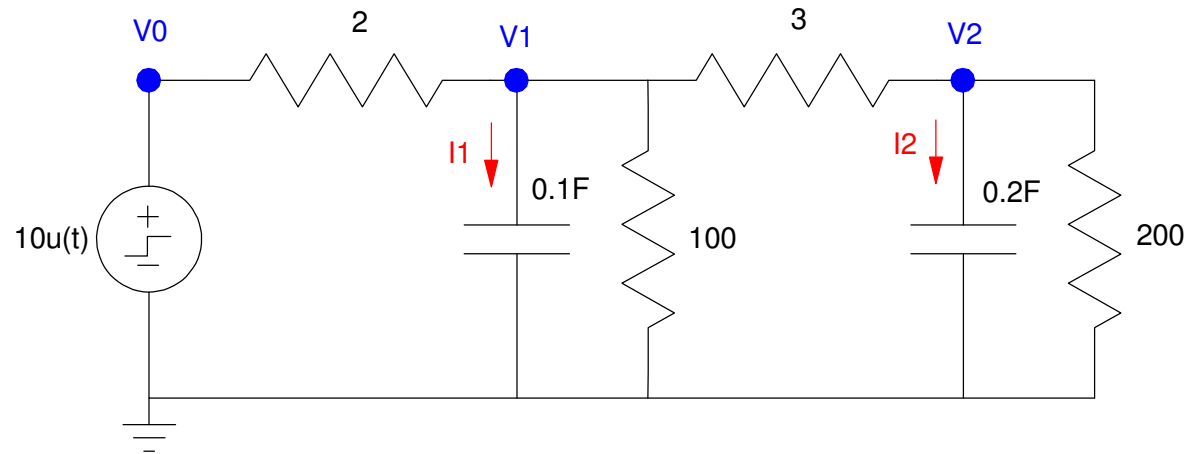


2-Stage RC Filter

The differential equations that describes this system are

$$\frac{dV_1}{dt} = 5V_0 - 8.433V_1 + 3.33V_2$$

$$\frac{dV_2}{dt} = 1.667V_1 - 1.692V_2$$



Rewrite in matrix form

$$\begin{bmatrix} \frac{dV_1}{dt} \\ \frac{dV_2}{dt} \end{bmatrix} = \begin{bmatrix} -8.433 & 3.33 \\ 1.667 & -1.692 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} V_0$$

$$\frac{dV}{dt} = AV + BV_0$$

In Matlab

A has eigenvectors and eigenvalues

```
>> A = [-8.4333  3.3333 ; 1.667 , -1.692]
```

```
    -8.4333    3.3333  
    1.6670   -1.6920
```

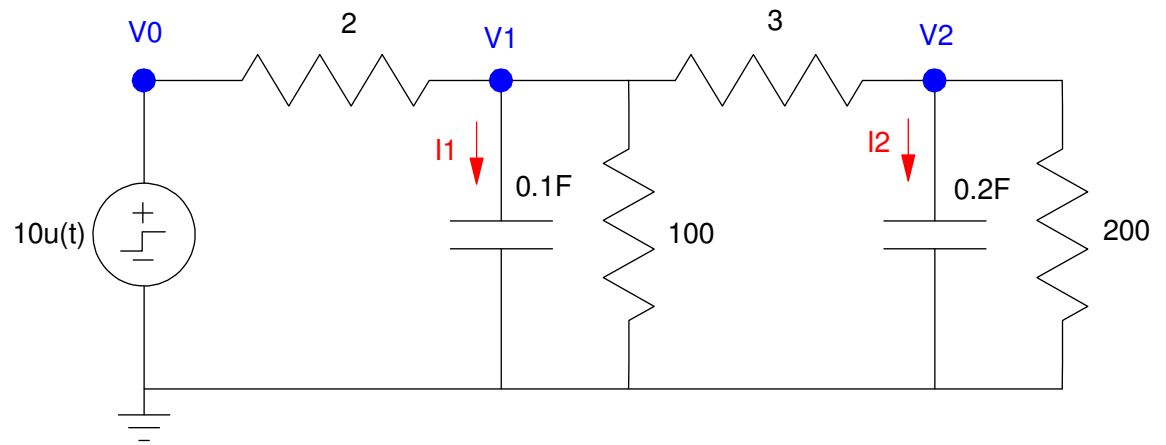
```
>> [Eigenvectors, Eigenvalues] = eig(A)
```

Eigenvectors =

```
   -0.9761   -0.4069  
    0.2174   -0.9135
```

Eigenvalues =

```
   -9.1758         0  
         0   -0.9495
```



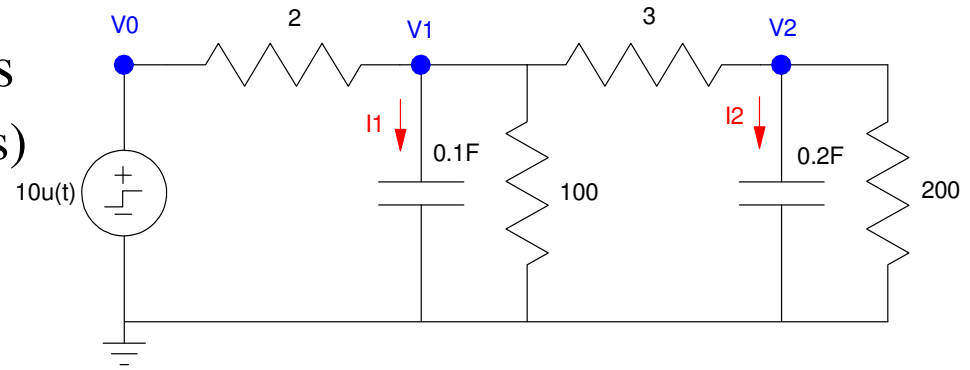
Eigenvalues

Eigenvalues tell you *how* the system behaves

- This is a second-order system (two capacitors)
- It has two eigenvalues: $\{-0.9495, -9.1758\}$

Give V1 & V2 an initial voltage

- The voltages decay according to the eigenvalues
- $V_1(t) = a_1 \cdot \exp(-0.9495t) + b_1 \cdot \exp(-9.1758t)$
- $V_2(t) = a_2 \cdot \exp(-0.9495t) + b_2 \cdot \exp(-9.1758t)$



Eigenvectors

Eigenvectors tell you *what* behaves that way

The natural response will be in the form of

$$V(t) = a \cdot \Lambda_1 \cdot \exp(\lambda_1 t) + b \cdot \Lambda_2 \cdot \exp(\lambda_2 t)$$

For this system

$$V(t) = a \cdot \begin{bmatrix} -0.9761 \\ 0.2174 \end{bmatrix} \cdot \exp(-9.1758t) + b \cdot \begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix} \cdot \exp(-0.9495t)$$

The initial condition tells you 'a' and 'b'

$$V(0) = a \cdot \begin{bmatrix} -0.9761 \\ 0.2174 \end{bmatrix} + b \cdot \begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix}$$

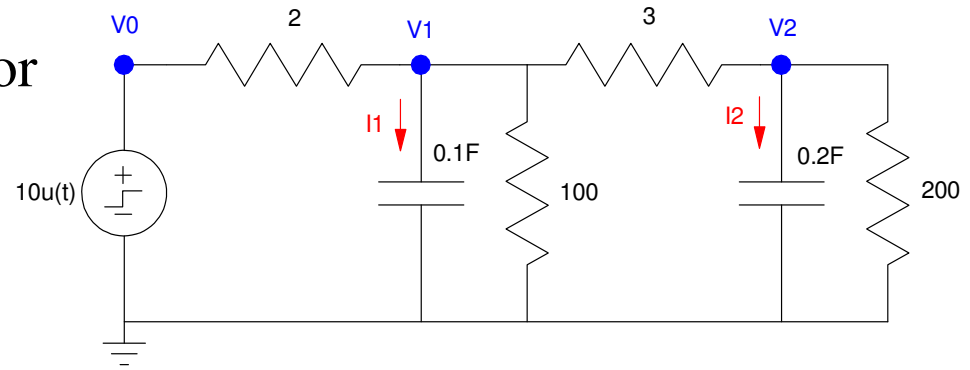
Fast Mode

Make the initial condition the fast eigenvector

$$V(0) = \begin{bmatrix} -0.9761 \\ 0.2174 \end{bmatrix}$$

Only the fast mode is excited

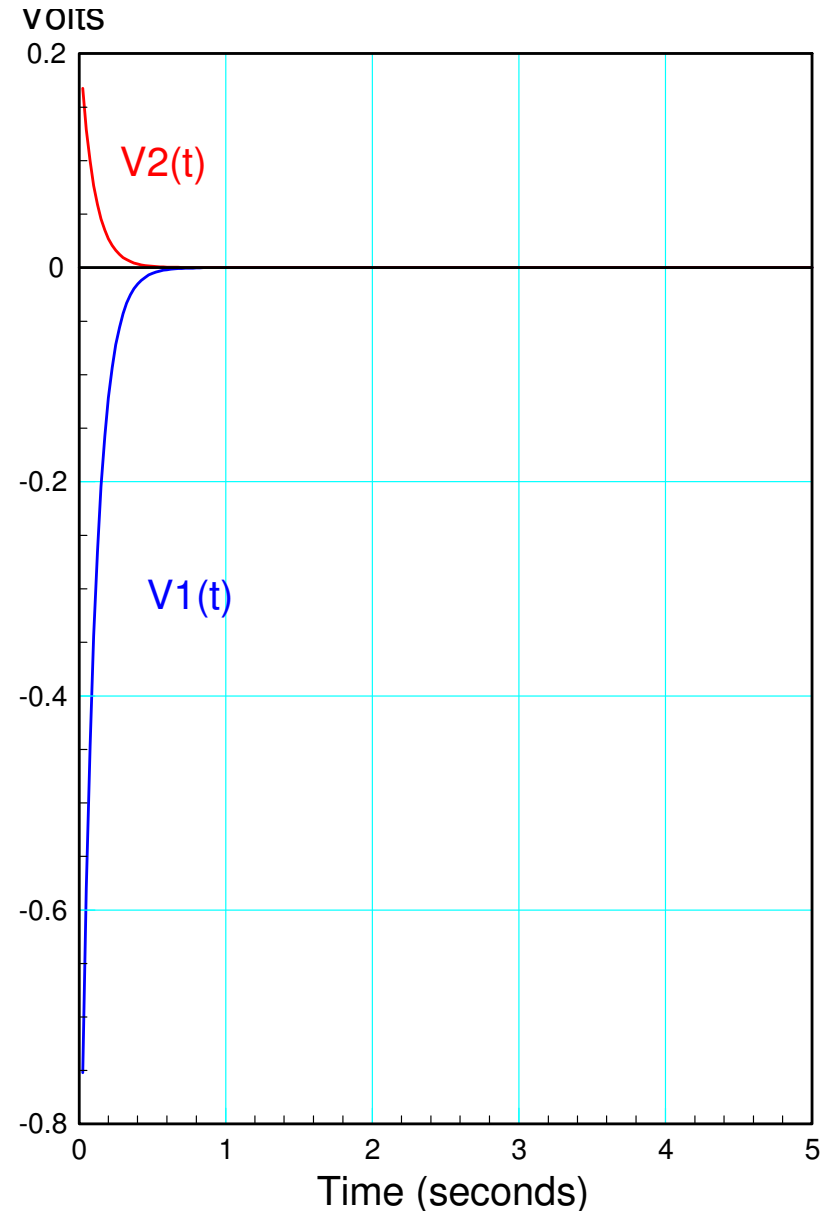
$$V(t) = 1 \cdot \begin{bmatrix} -0.9761 \\ 0.2174 \end{bmatrix} \cdot \exp(-9.1758t) + 0 \cdot \begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix} \cdot \exp(-0.9495t)$$



Fast Mode

Matlab Code

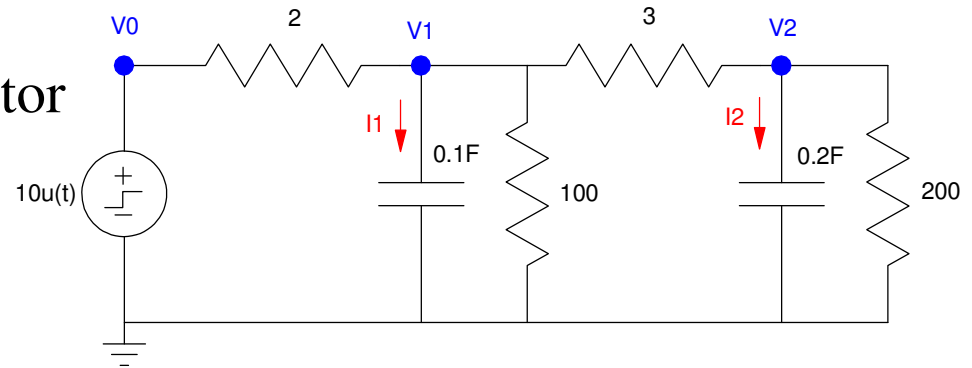
```
V1 = -0.9761;  
V2 = 0.2174;  
V0 = 0;  
t = 0;  
dt = 0.025  
y = [];  
while (t < 5)  
    dV1 = 5*V0 - 8.433*V1 + 3.33*V2;  
    dV2 = 1.667*V1 - 1.692*V2;  
  
    V1 = V1 + dV1*dt;  
    V2 = V2 + dV2*dt;  
    t = t + dt;  
    y = [y ; t, V0, V1, V2];  
end  
  
plot(y(:,1),y(:,2:4))
```



Slow Mode

Make the initial condition the slow eigenvector

$$V(0) = \begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix}$$



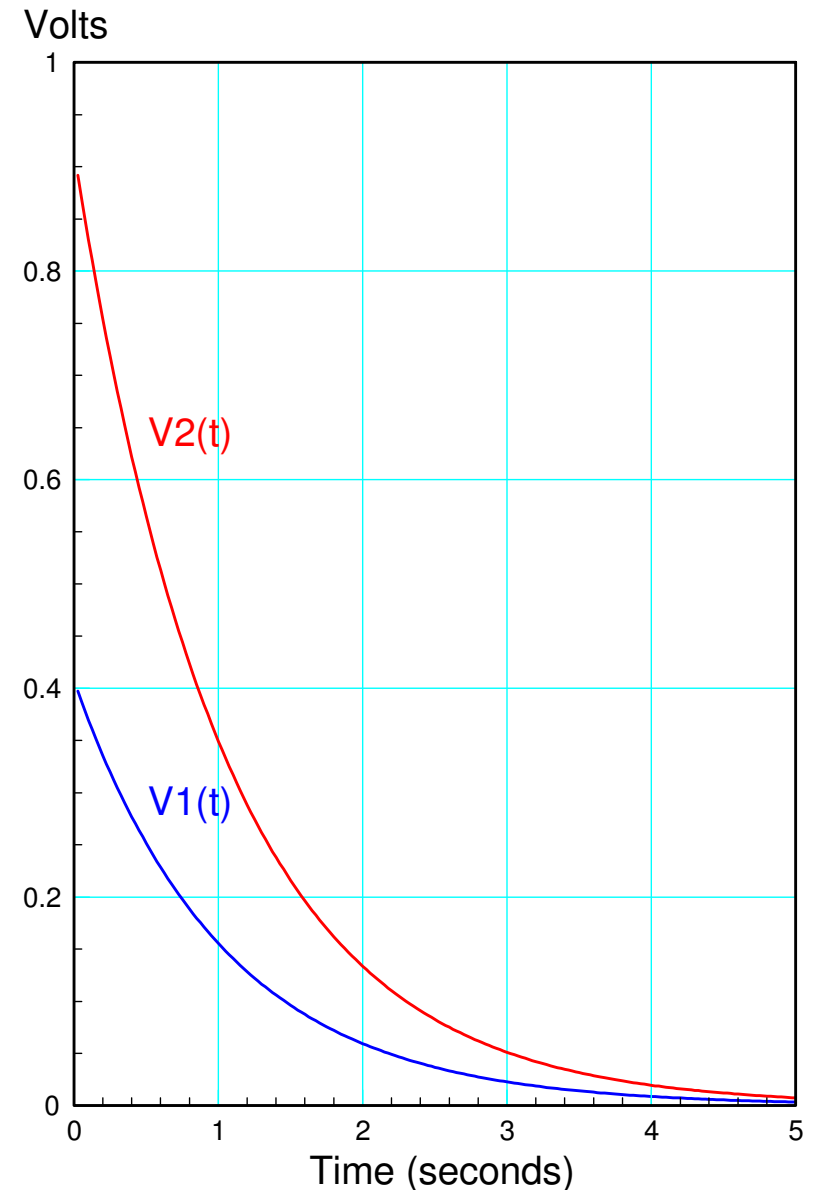
Only the slow mode is excited

$$V(t) = 0 \cdot \begin{bmatrix} -0.9761 \\ 0.2174 \end{bmatrix} \cdot \exp(-9.1758t) + 1 \cdot \begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix} \cdot \exp(-0.9495t)$$

Slow Mode

Matlab Code

```
V1 = 0.4069;  
V2 = 0.9135;  
V0 = 0;  
t = 0;  
dt = 0.025  
y = [];  
while(t < 5)  
    dV1 = 5*V0 - 8.433*V1 + 3.33*V2;  
    dV2 = 1.667*V1 - 1.692*V2;  
  
    V1 = V1 + dV1*dt;  
    V2 = V2 + dV2*dt;  
    t = t + dt;  
    y = [y ; t, V0, V1, V2];  
end  
  
plot(y(:,1),y(:,2:4))
```



Fast & Slow Modes

In general, both modes are excited

$$V(t) = a \cdot \begin{bmatrix} -0.9761 \\ 0.2174 \end{bmatrix} \cdot \exp(-9.1758t) + b \cdot \begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix} \cdot \exp(-0.9495t)$$

$$V(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

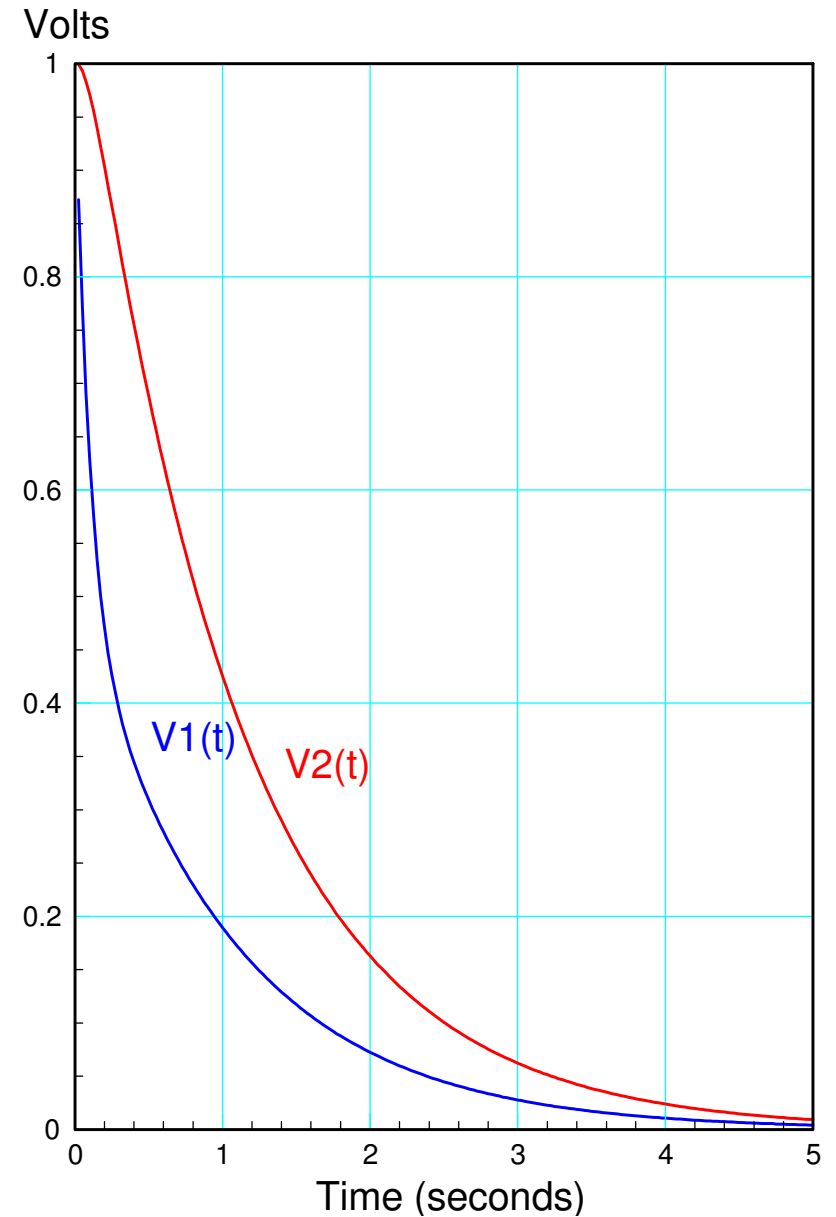
Both modes are excited

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = a \cdot \begin{bmatrix} -0.9761 \\ 0.2174 \end{bmatrix} + b \cdot \begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -0.5169 \\ 1.2177 \end{bmatrix}$$

In Matlab

```
V1 = 1;  
V2 = 1;  
V0 = 0;  
t = 0;  
dt = 0.025  
y = [];  
while(t < 5)  
    dV1 = 5*V0 - 8.433*V1 + 3.33*V2;  
    dV2 = 1.667*V1 - 1.692*V2;  
  
    V1 = V1 + dV1*dt;  
    V2 = V2 + dV2*dt;  
    t = t + dt;  
    y = [y ; t, V0, V1, V2];  
end  
  
plot(y(:,1),y(:,2:4))
```



Summary

Capacitors are integrators

$$I = C \frac{dV}{dt}$$

$$V = \frac{1}{C} \int I(t) \cdot dt$$

Each capacitor adds a 1st-order differential equation to a circuit

To find the voltages with an RC circuit

- Use CircuitLab's time response, or
- Use numerical integration

Eigenvalues & eigenvectors tell you

- *How* the system behaves (eigenvalues)
- *What* behaves that way (eigenvectors)

