Inductors

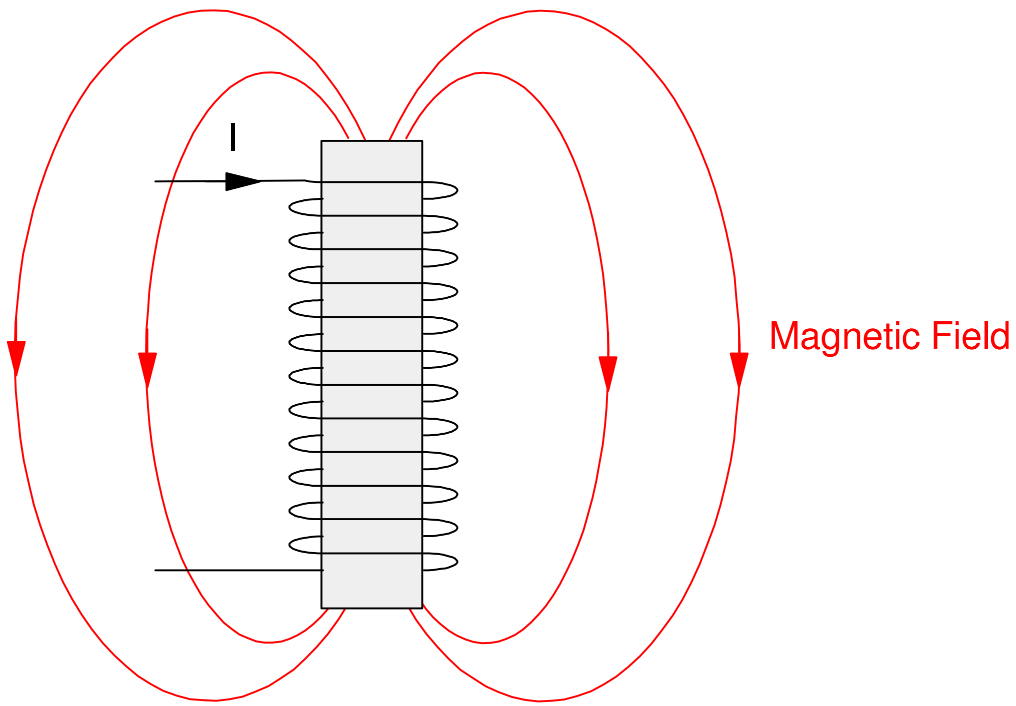
ECE 211 Circuits I

Lecture #19

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Inductors

- Capacitors deal with voltage and store energy in an electric field.
- Inductors deal with current and store energy in a magnetic field.



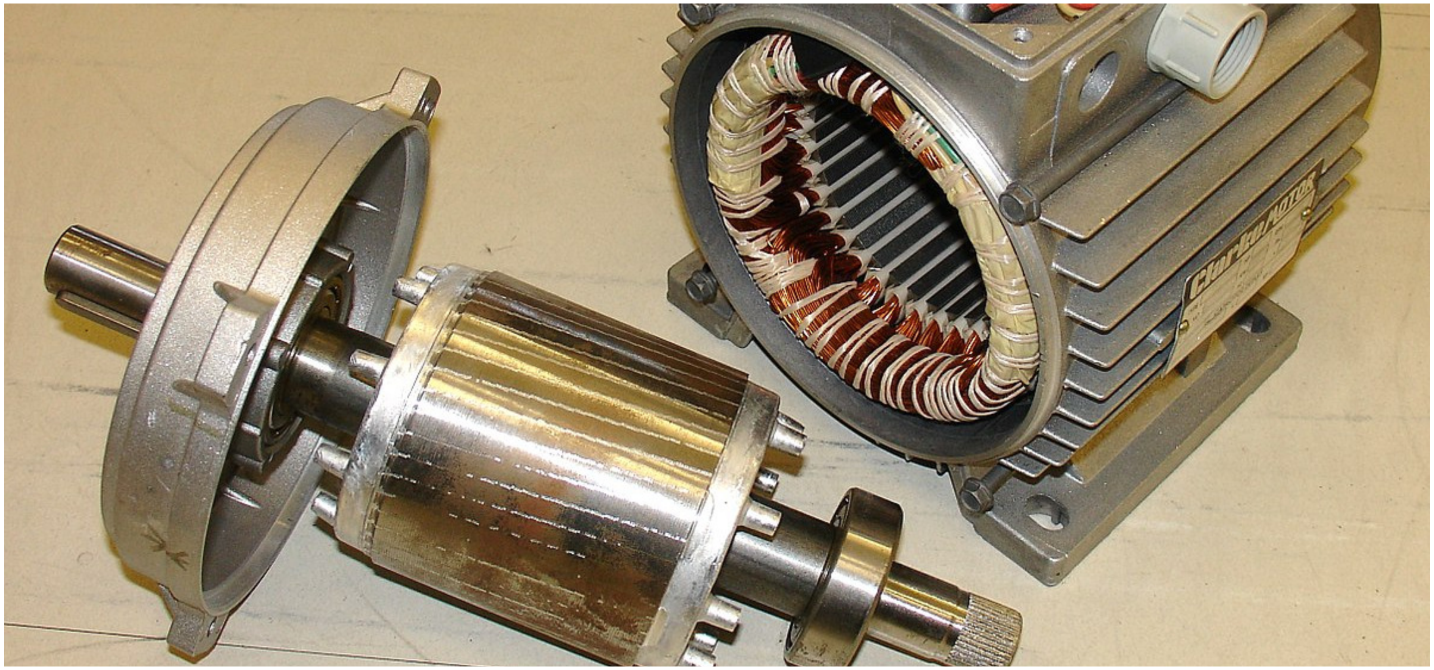
Inductors

In general, inductors are avoided if at all possible:

- Unlike electric fields and capacitors, magnetic fields do not have starting point and an ending point. That means that the magnetic fields extend outside the inductor and affect other components in a circuit.
- The core of an inductor almost has to be iron. Iron is about the only material which is strongly magnetic. This dependence upon iron makes inductors heavy, lossy, and only capable of supporting a limited amount of current until the iron saturates.
- The large amount of windings necessary to make an inductor creates a large resistive element with inductors, also reducing the efficiency.

Sometimes, you just can't avoid using inductors:

- Speakers rely upon electromagnets to work. An electromagnet is inherently an inductor.
- Motors rely upon electromagnetics to work. Again, any time you have an electromagnet you inherently have an inductor.
- Transmission lines are wires that carry electricity large distances. Current through a wire creates a magnetic field. This magnetic field stores energy, and again, created inductance.



Example 1: Determine R & L

- 100 windings of 36 gage wire copper wire (1.38 Ohms / meter)[1]
- Cross sectional area = $25mm^2$
- Iron core with relative permeability of 800

Solution: From Electronics Tutorials[2]

$$L = \mu N^2 A / l = 0.0101H$$

$$R = (100 \text{ windings})\left(0.02\frac{\text{m}}{winding}\right)(1.38\tfrac{\Omega}{m}) = 2.77\Omega$$

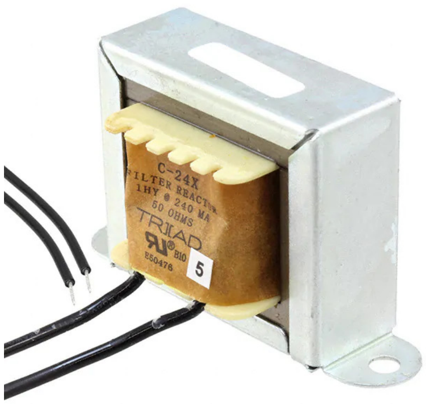
You can't avoid resistance with inductors

1 https://www.engineeringtoolbox.com/copper-wire-d_1429.html

2 https://www.electronics-tutorials.ws/inductor/inductor.html

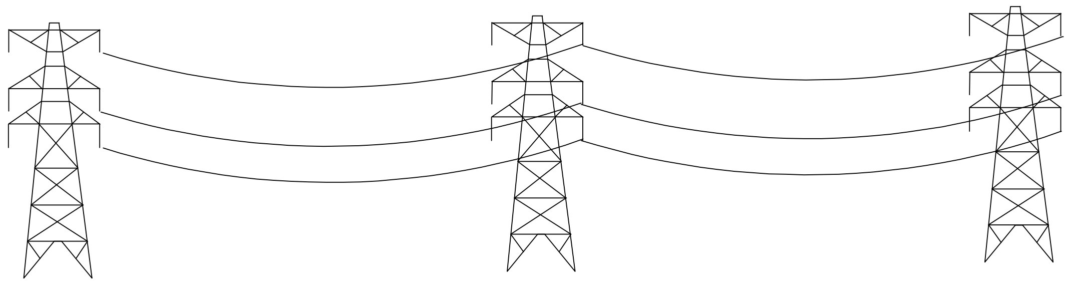
This shows up in inductors you can get from Digikey:

- If you want large inductance, you have high resistance
- If you want low resistance, you get low inductance
- If you want a small inductor, you get both: low inductance and high resistance

CX24	5250-RC
	
1 H	0.0001H
50 Ohms	0.216 Ohms
240mA (max)	2 A
$10.23 ea	$2.11

Example 2: Determine the inductance of a copper transmission line:

- Length = 1km
- radius = 1cm
- frequency = 60Hz



From Wikipedia[3]

$$L = \frac{\mu_0}{2\pi} l(A - B + C) = 2.30\,\frac{mH}{km}$$

$$A = \ln\left(\frac{l}{r} + \sqrt{\left(\frac{l}{r}\right)^2 + 1}\right) = 12.20$$

$$B = \frac{1}{\frac{r}{l} + \sqrt{1 + \left(\frac{r}{l}\right)^2}} \approx 1$$

$$C = \frac{1}{4 + r\sqrt{\frac{2}{\rho}\omega\mu}} = 0.1569$$

3 https://en.wikipedia.org/wiki/Inductor#Inductance_formulas

VI Characteristics for an Inductor

The energy stored in an inductor is

$$E = \frac{1}{2}LI^2$$

Power is the derivative of energy

$$P = VI = \frac{d}{dt}\left(\frac{1}{2}LI^2\right)$$

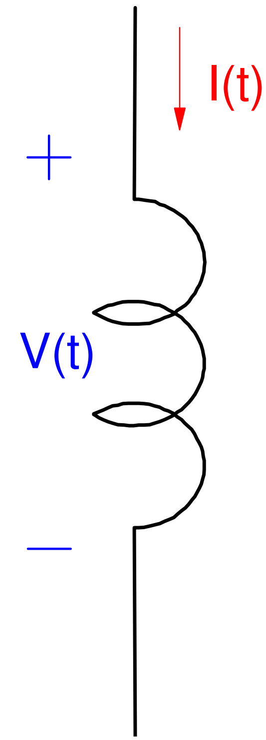
$$VI = LI \cdot \frac{dI}{dt}$$

Giving

$$\boxed{V = L\frac{dI}{dt}}$$

Meaning

- Each inductor adds a 1st-order differential equation to a circuit
- The differential equation is in terms of current.



Single Inductor Circuit

1st-order differential equation in current

$$V_0(t) = 10\,u(t)$$

$$V_1 = 4I_L$$

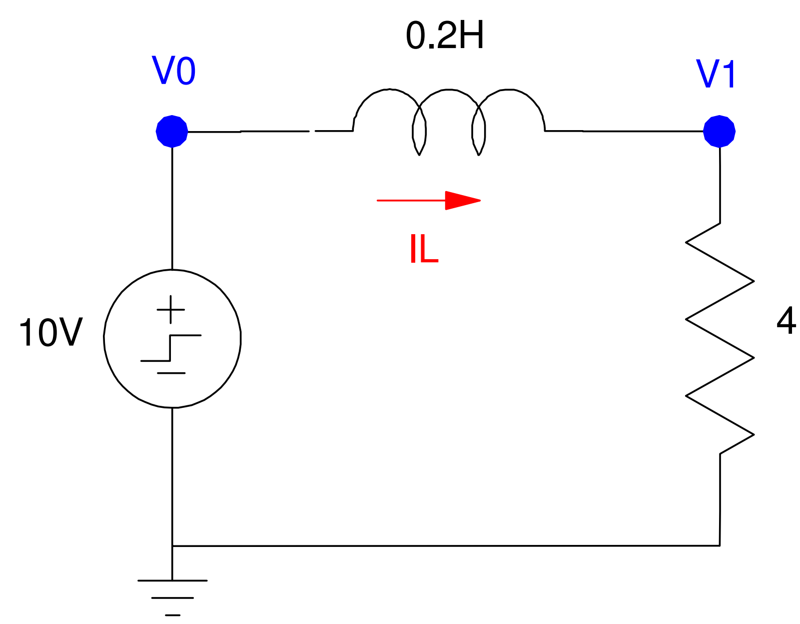
$$V_L = 0.2\frac{dI_1}{dt} = V_0 - V_1$$

Substituting

$$0.2\frac{dI_L}{dt} = V_0 - 4I_L$$

or

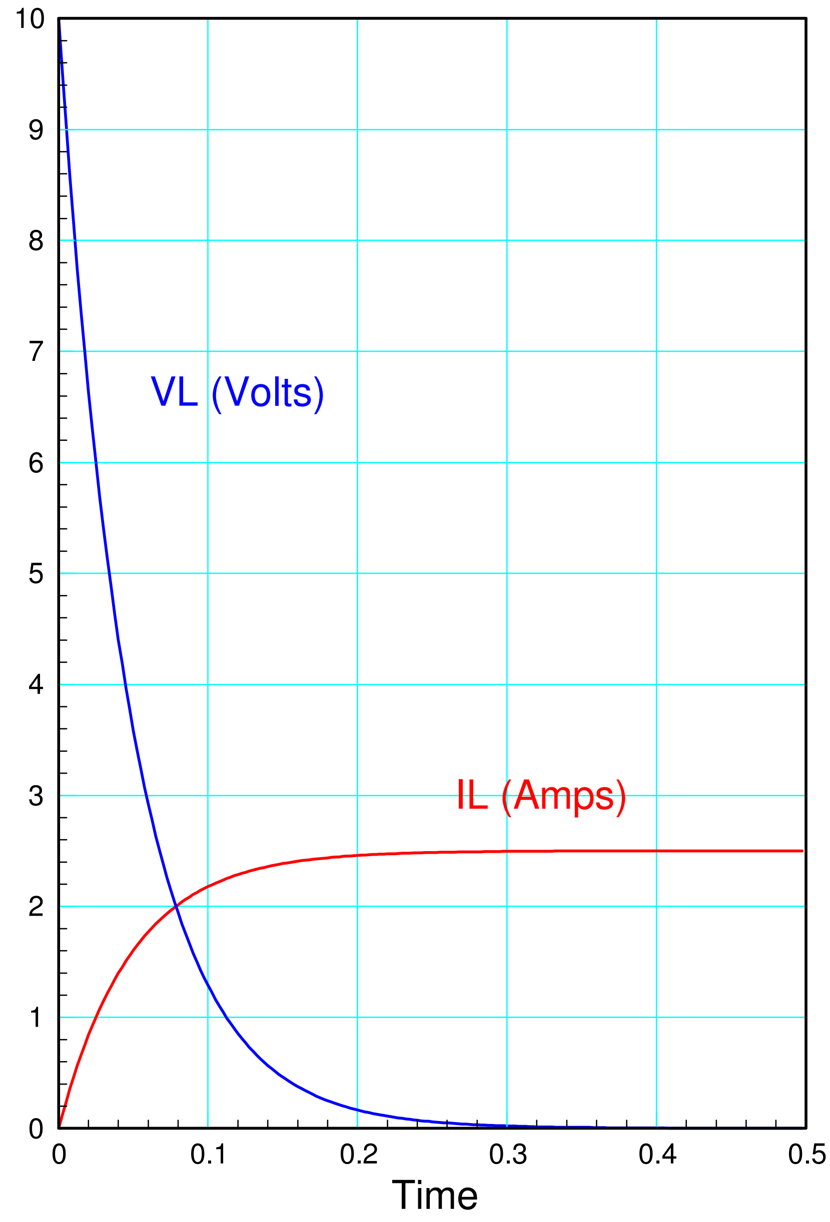
$$\boxed{\frac{dI_L}{dt} = 5V_0 - 20I_L}$$



Matlab Solution

- Solve using numerical integration like we did with capacitors:

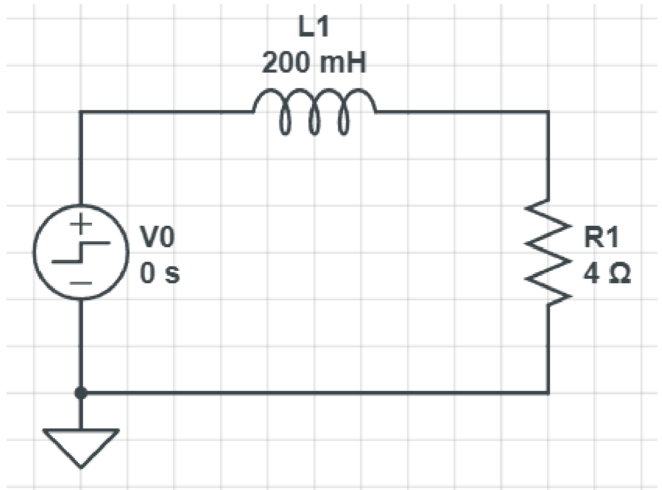
```
V0 = 10;
IL = 0
dt = 0.0025;
t = 0;
y = [];

while(t < 0.5)
    VL = V0 - 4*IL;
    y = [y ; [t, IL, VL] ];
    dIL  = 5*V0 - 20*IL;
    IL = IL + dIL * dt;
    t = t + dt;

    end

plot(y(:,1), y(:,2:3))
```

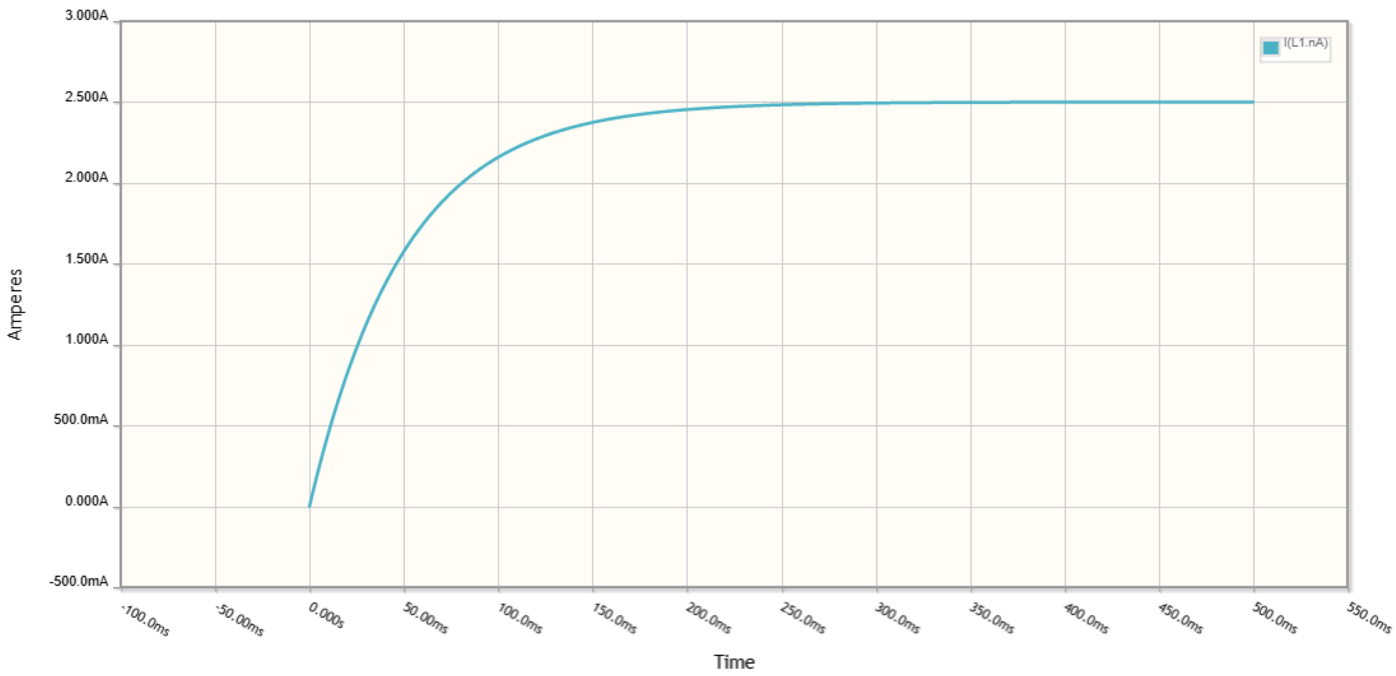


CircuitLab Solution

Time-Domain simulation

- Same results as Matlab
- CircuitLab also uses numerical integration to solve



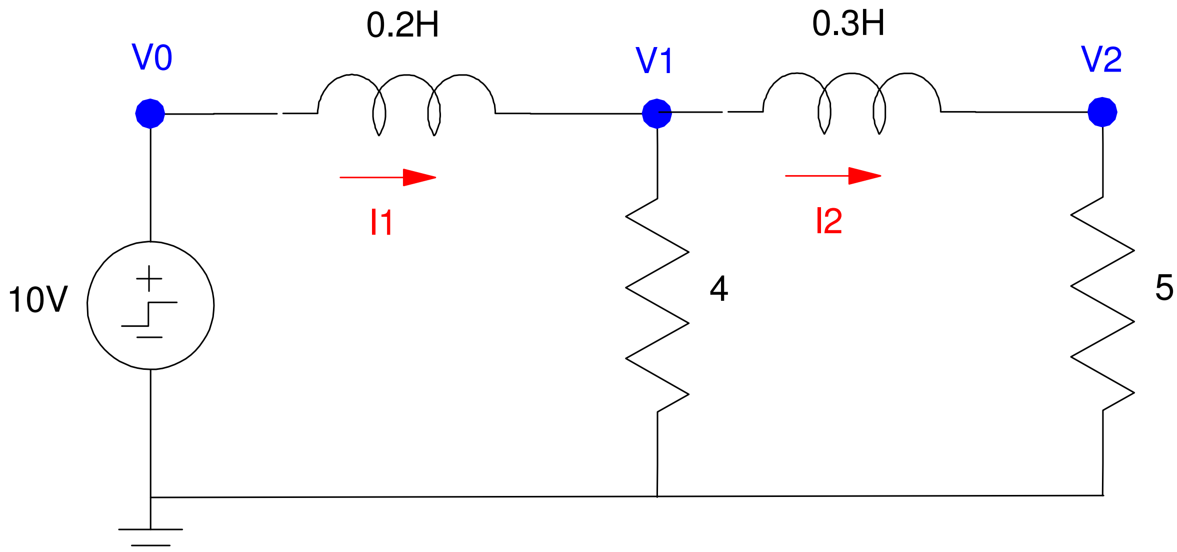


Two Inductor Circuit

- Two inductors mean two ways to store energy
- 2nd-order differential equation

Write the differential equations that describe this circuit

- Several ways to do this...



Two Indictor Circuit

Find V1 and V2 in terms if I1 and I2

$$V_1 = 4(I_1 - I_2)$$

$$V_2 = 5I_2$$

Use the inductor equations

$$0.2\frac{dI_1}{dt} = V_0 - V_1$$

$$0.3\frac{dI_2}{dt} = V_1 - V_2$$

Substituting

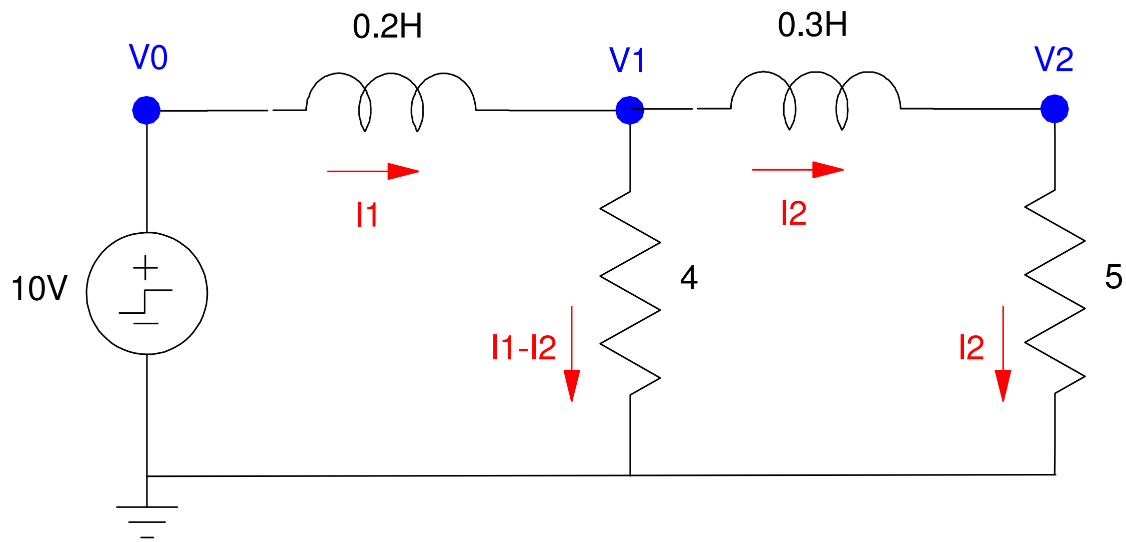
$$0.2\frac{dI_1}{dt} = V_0 - 4(I_1 - I_2)$$

$$0.3\frac{dI_2}{dt} = 4(I_1 - I_2) - (5I_2)$$

Simplify

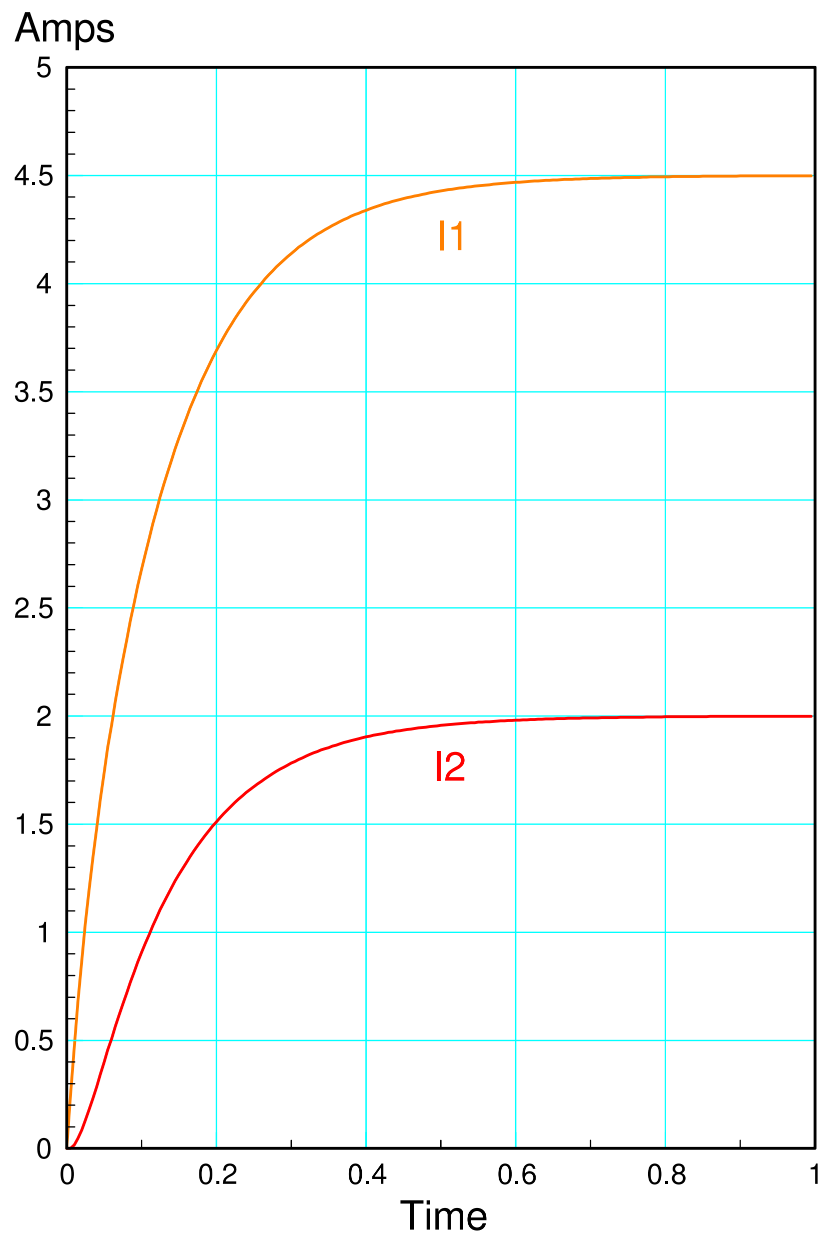
$$\frac{dI_1}{dt} = 5V_0 - 20I_1 + 20I_2$$

$$\frac{dI_2}{dt} = 13.333I_1 - 30I_2$$



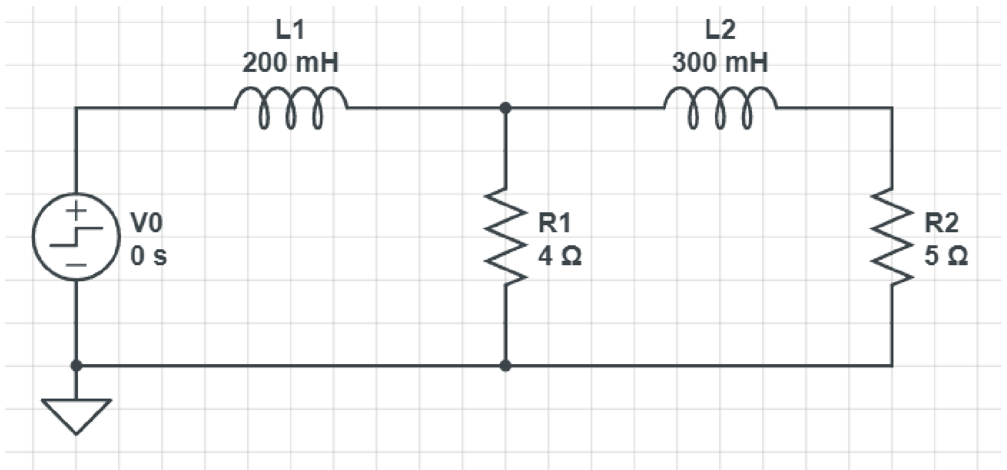
Solve in Matlab

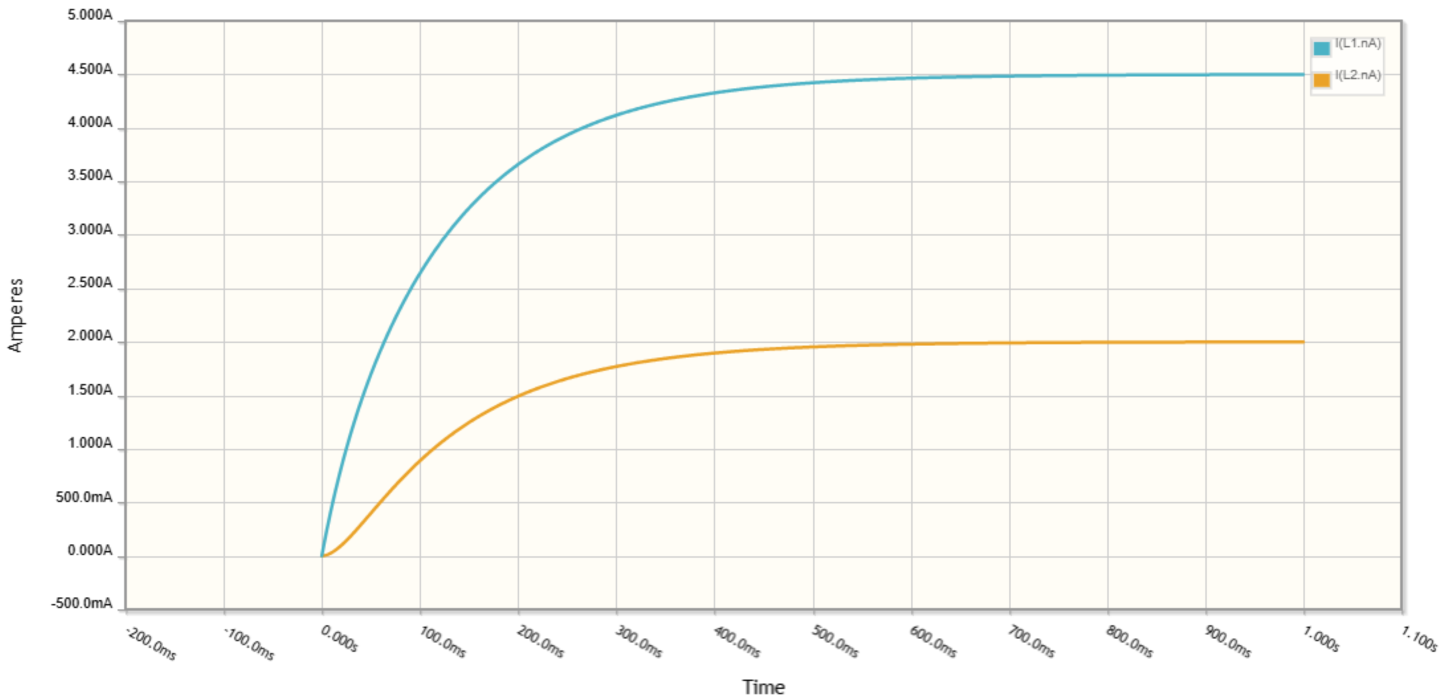
```
V0 = 10;
I1 = 0;
I2 = 0;
dt = 0.005;
t = 0;
y = [];

while(t < 1)
   y = [y ; [t, I1, I2] ];
   dI1 = 5*V0 - 20*I1 + 20*I2;
   dI2 = 13.333*I1 - 30*I2;
   I1 = I1 + dI1*dt;
   I2 = I2 + dI2*dt;
   t = t + dt;
   end

plot(y(:,1), y(:,2:3))
```



CircuitLab Solution

- Time-domain simulation
- Same results as Matlab
- CircuitLab also uses numerical methods

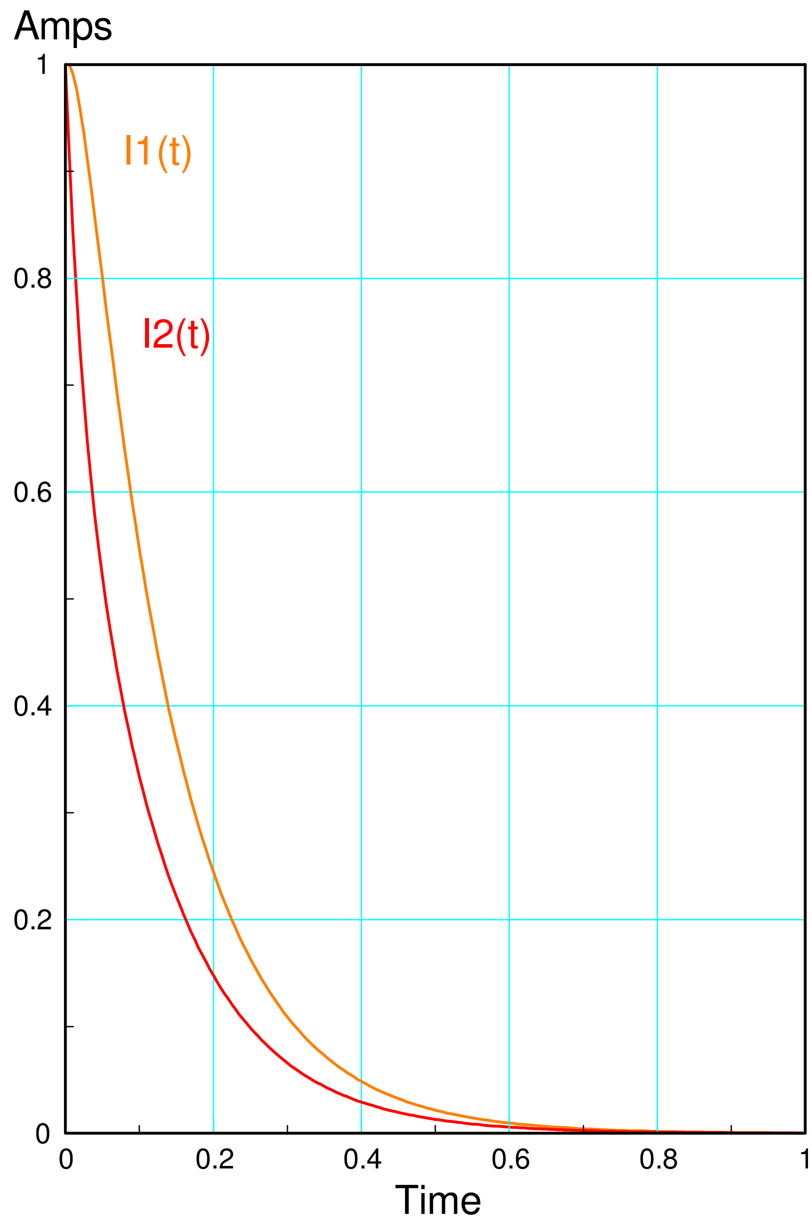




Natural Response

- Set V0 = 0
- Set initial conditions on I1 and I2

```
V0 = 0;
I1 = 1;
I2 = 1;
dt = 0.005;
t = 0;
y = [];

while(t < 1)
   y = [y ; [t, I1, I2] ];
   dI1 = 5*V0 - 20*I1 + 20*I2;
   dI2 = 13.333*I1 - 30*I2;
   I1 = I1 + dI1*dt;
   I2 = I2 + dI2*dt;
   t = t + dt;
   end

plot(y(:,1), y(:,2:3))
```



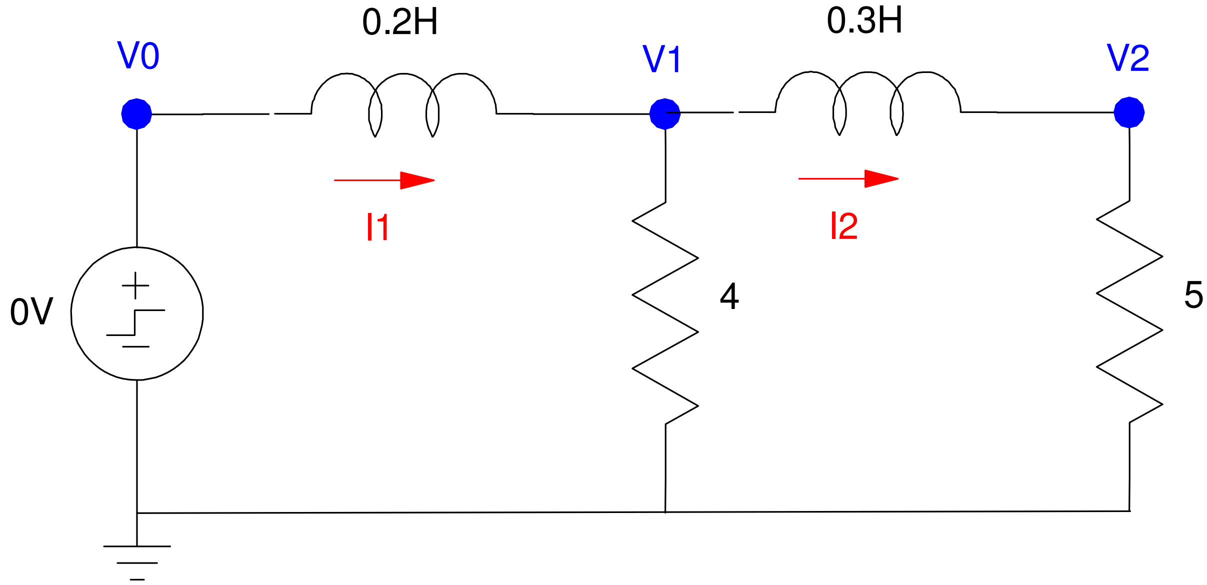
How Fast? How Slow?

Assume

- $V0 = 0$
- $I1(0) = 1.$

What should $I2(0)$ be so that $I1(t)$

- Decays as fast as possible?
- Decays as slow as possible?
- How fast is that?



How Fast?

This is an eigenvalue problem

The dynamics are:

$$\frac{dI_1}{dt} = 5V_0 - 20I_1 + 20I_2$$

$$\frac{dI_2}{dt} = 13.333I_1 - 30I_2$$

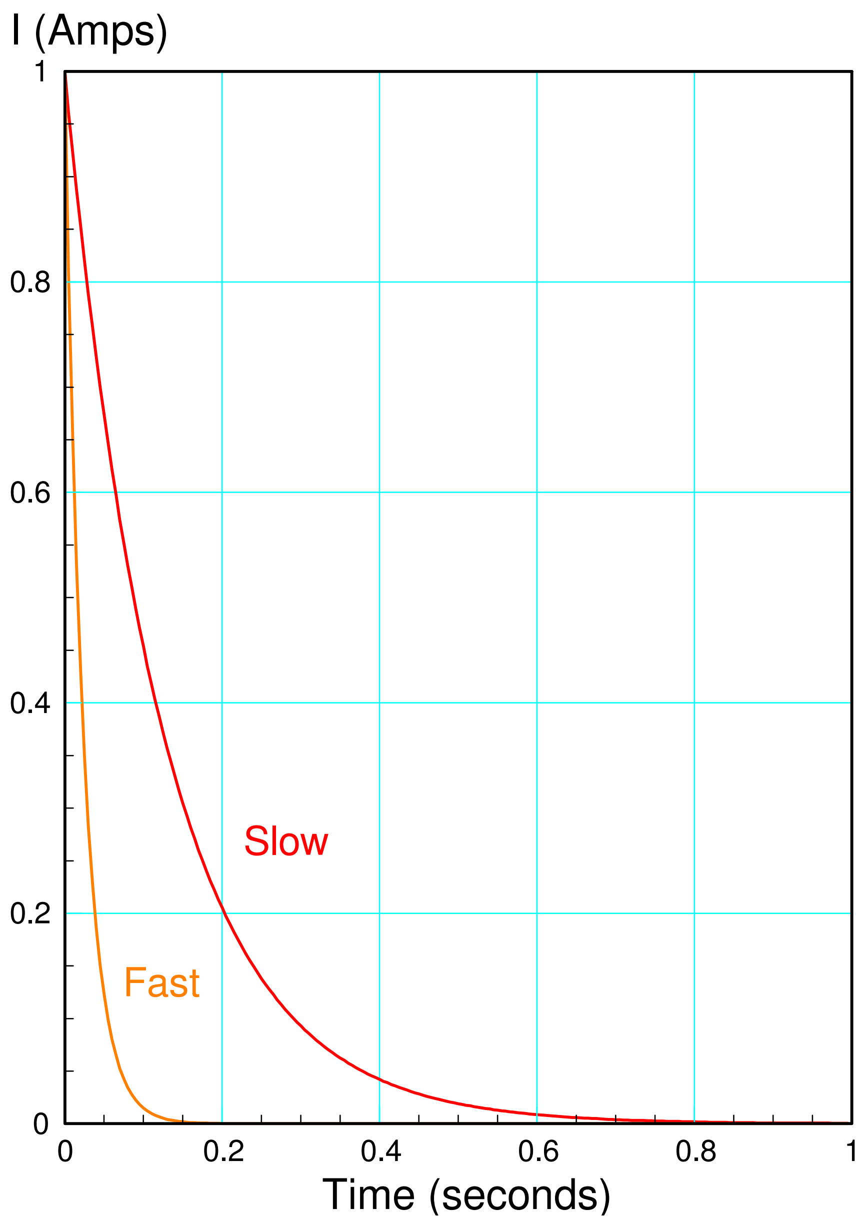
Rewrite in matrix form

$$\begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = \begin{bmatrix} -20 & 20 \\ 13.333 & -30 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} V_0$$

A has two eigenvalues:

- This is how V1(t) decays

```
>> A = [-20,20 ; 13.333,-30];
>> eig(A)
   -7.9219
  -42.0781
```



What is the fast mode?

This is an eigenvetor problem

- A has two eigenvalues

```
>> eig(A)

    slow       fast
   -7.9219  -42.0781
```

- A has two eigenvectors

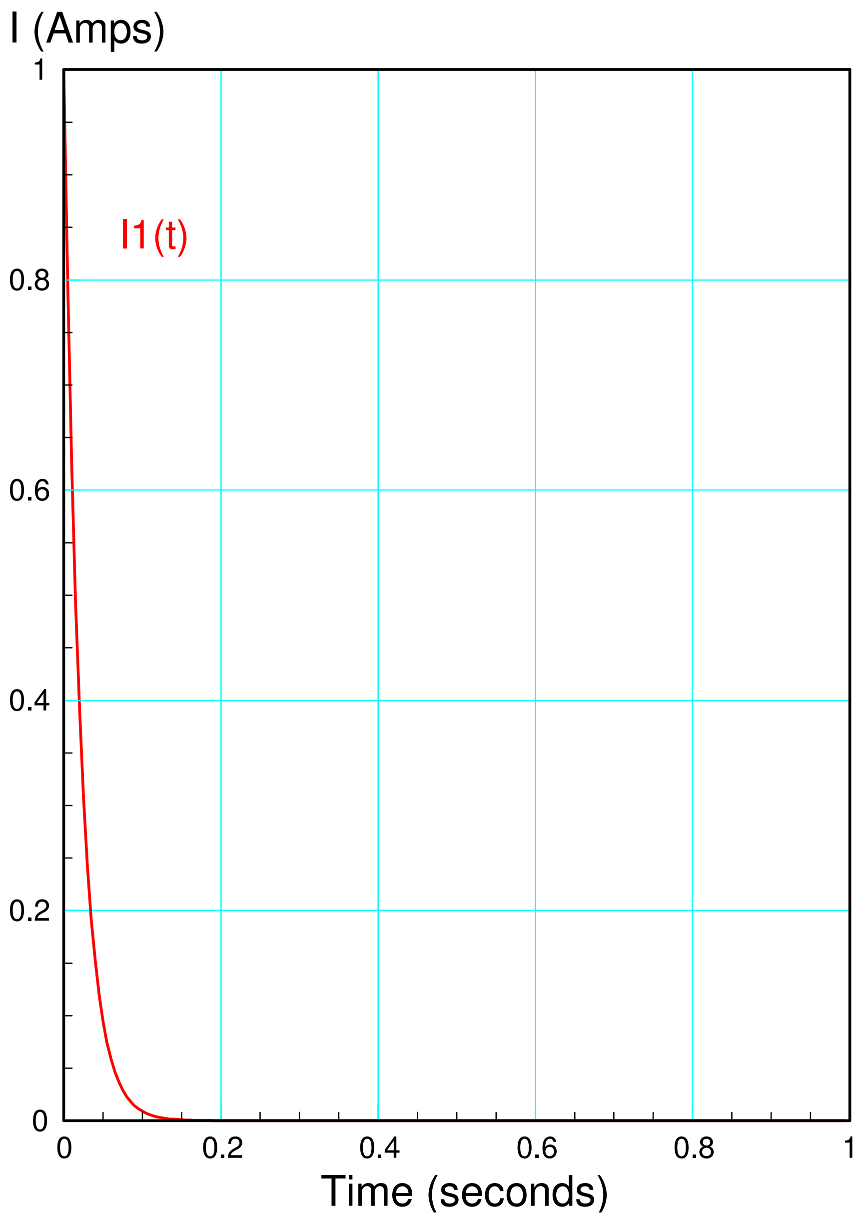
```
>> [M,V] = eig(A)

    slow       fast
    0.8560   -0.6714
    0.5170    0.7411
```

To decay as fast as possible, make the initial condition the fast eigenvector

$$I(0) = \alpha \begin{bmatrix} -0.6714 \\ 0.7414 \end{bmatrix}$$

Fast Mode in Matlab

```
V0 = 0;
I1 = -0.6714 / -0.6714;
I2 = 0.7411  / -0.6714;
dt = 0.005;
t = 0;
y = [];

while(t < 1)
   y = [y ; [t, I1] ];
   dI1 = 5*V0 - 20*I1 + 20*I2;
   dI2 = 13.333*I1 - 30*I2;
   I1 = I1 + dI1*dt;
   I2 = I2 + dI2*dt;
   t = t + dt;
   end

plot(y(:,1), y(:,2))
```



What is the slow mode?

A has two eigenvalues

```
>> eig(A)

     slow        fast
   -7.9219   -42.0781
```

A has two eigenvectors

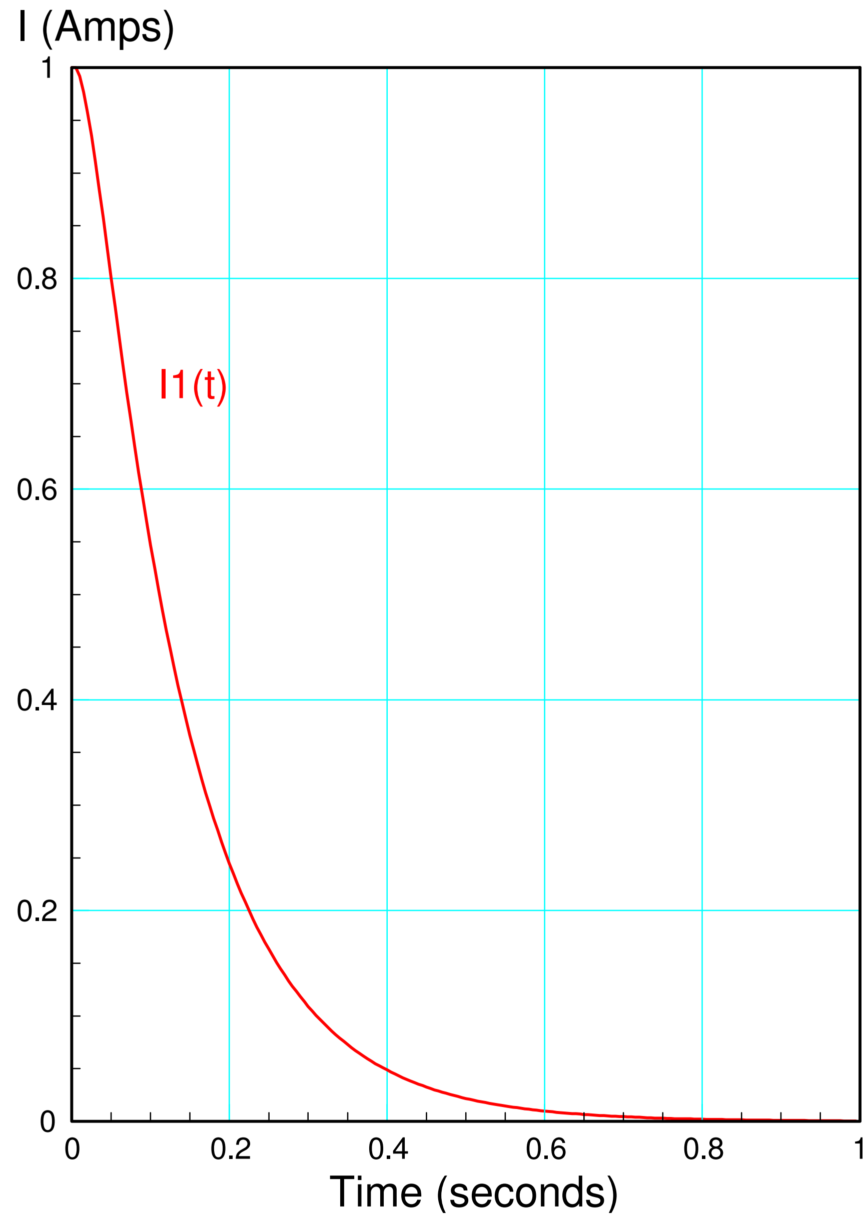
```
>> [M,V] = eig(A)

     slow        fast
     0.8560    -0.6714
     0.5170     0.7411
```

To decay as slowt as possible, make the initial condition the slow eigenvector

$$I(0) = \beta \begin{bmatrix} 0.8560 \\ 0.5170 \end{bmatrix}$$

Slow Mode in Matlab

```
V0 = 0;
I1 = 0.8560 / 0.8560;
I2 = 0.5170 / 0.5170;
dt = 0.005;
t = 0;
y = [];

while(t < 1)
   y = [y ; [t, I1] ];
   dI1 = 5*V0 - 20*I1 + 20*I2;
   dI2 = 13.333*I1 - 30*I2;
   I1 = I1 + dI1*dt;
   I2 = I2 + dI2*dt;
   t = t + dt;
   end

plot(y(:,1), y(:,2))
```



Both Modes

An arbitrary initial condition excites both modes

$$I(t) = \alpha \begin{bmatrix} -0.6714 \\ 0.7414 \end{bmatrix} \cdot \exp(-42.08t) + \beta \begin{bmatrix} 0.8560 \\ 0.5170 \end{bmatrix} \cdot \exp(-7.92t)$$

$$I(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} -0.6714 \\ 0.7414 \end{bmatrix} + \beta \begin{bmatrix} 0.8560 \\ 0.5170 \end{bmatrix}$$

Solving

$$\alpha = -0.06$$

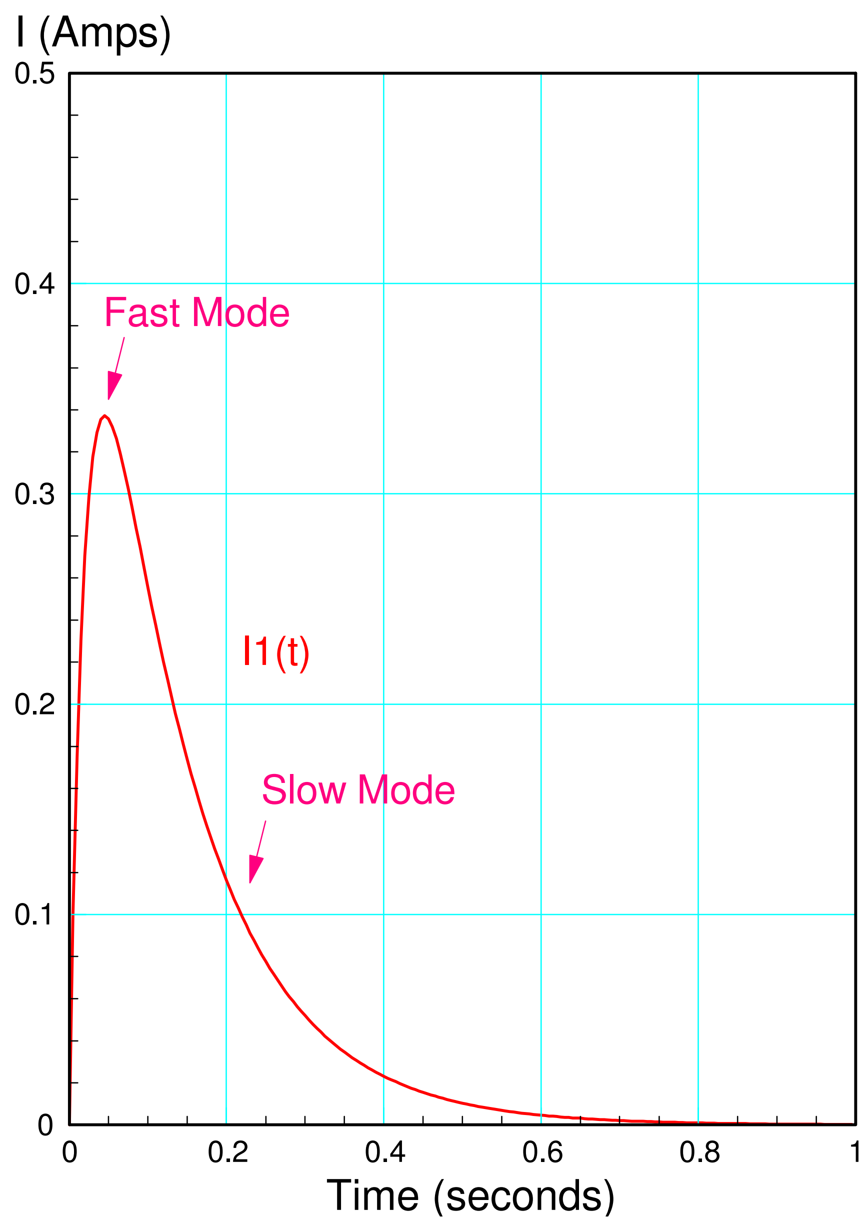
$$\beta = -0.06$$

In Matlab

Both modes show up in I1(t)

- The fast mode decays quickly
- Leaving the slow mode

```
V0 = 0;
I1 = 0;
I2 = 1;
dt = 0.005;
t = 0;
y = [];

while(t < 1)
   y = [y ; [t, I1] ];
   dI1 = 5*V0 - 20*I1 + 20*I2;
   dI2 = 13.333*I1 - 30*I2;
   I1 = I1 + dI1*dt;
   I2 = I2 + dI2*dt;
   t = t + dt;
   end

plot(y(:,1), y(:,2))
```



Voltage Amplification

- New topic

With inductors, you can get more voltage out than you put in

- If the current suddenly goes to zero, the voltage goes to infinity
- This is how alternators and spark plugs work in your car.

$$V = L\frac{dI}{dt}$$

Voltage Amplification Example

- Let R varies from 5 Ohms to 1000 Ohms (switch closed or switch open)
- This also illustrates voltage inversion (-12V produces +2000V)

Write the differential equations

$$\left(\frac{V_1 - V_0}{R}\right) + \left(\frac{V_1}{100}\right) + I_1 = 0$$

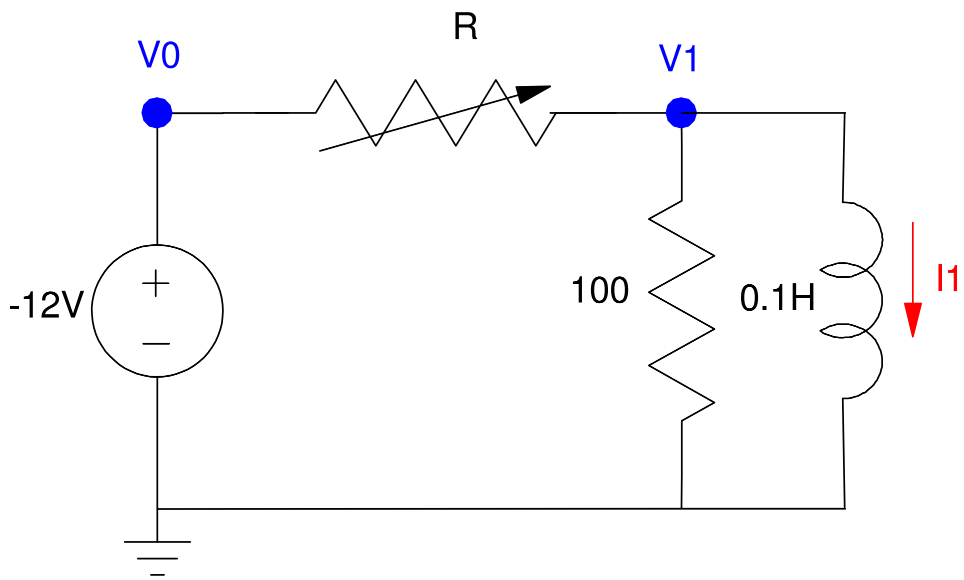
$$V_1 = L\frac{dI_1}{dt}$$

giving

$$\left(\frac{1}{R} + \frac{1}{100}\right)V_1 = \left(\frac{V_0}{R}\right) - I_1$$

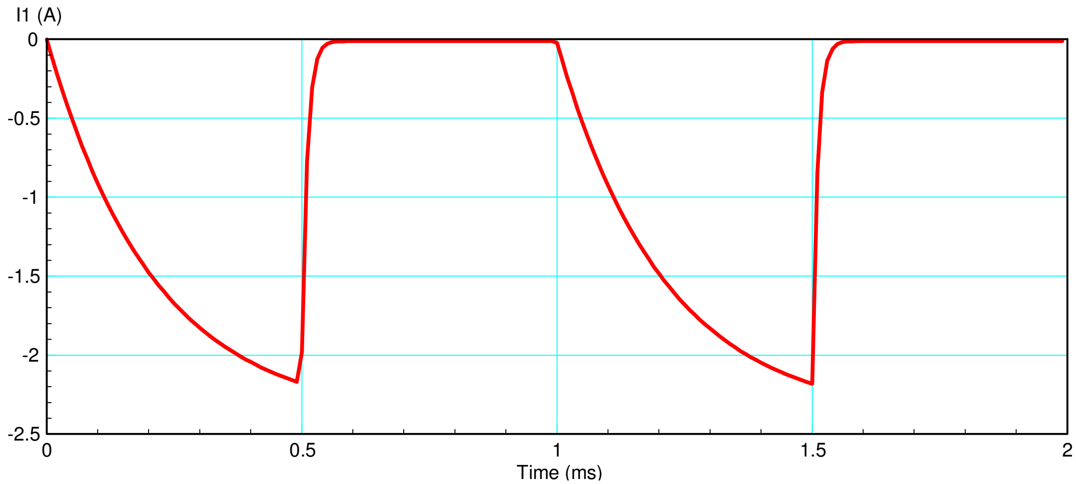
$$\left(\frac{1}{R} + \frac{1}{100}\right)\left(\frac{dI_1}{dt}\right) = \left(\frac{V_0}{R}\right) - I_1$$

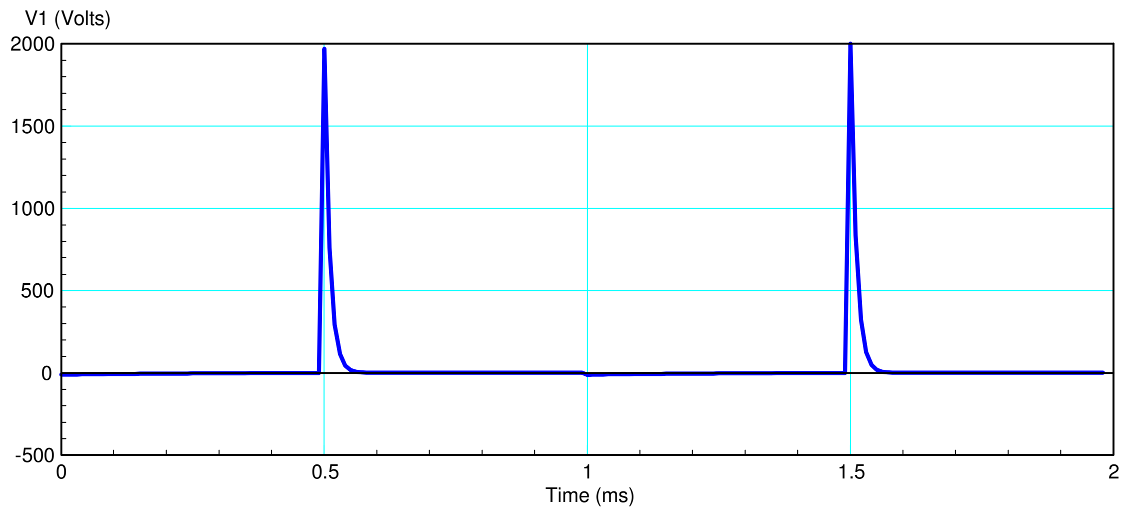
$$\frac{dI_1}{dt} = \left(\frac{1}{R} + \frac{1}{100}\right)^{-1}\left(\left(\frac{V_0}{R}\right) - I_1\right)$$



Solve in Matlab:

```
V0 = -12;
I = 0
dt = 0.001;
t = 0;
Y = [];

while(t < 2)
   if(sin(2*pi*t) > 0)
       R = 5;
   else
       R = 1000;
   end

   dI=(V0/R-I)/(1/R+1/100);

   I = I + dI * dt;
   t = t + dt;
   V1 = V0 - I*R;

   Y = [Y ; [I, V1] ];
end
```





Summary

Inductors store energy in a magnatic field

$$E = \frac{1}{2} L I^2$$

The VI characteristics for an inductor are

$$V = L \frac{dI}{dt}$$

Each inductor in a circuit adds a 1st-order differential equation

The resulting currents can be solved using

- CircuitLab, or
- Matlab with numerical integration, or
- Calculus (coming later)