
Phasor Voltages

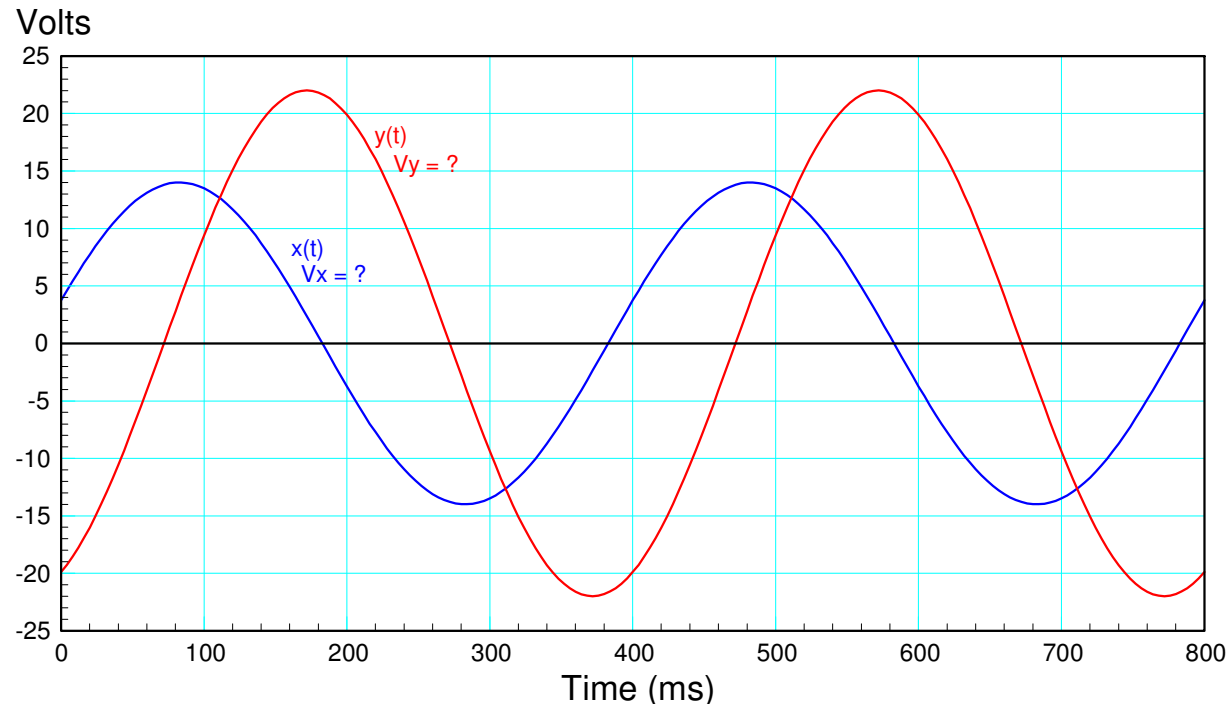
ECE 211 Circuits I

Lecture #22

Please visit [Bison Academy](#) for corresponding lecture notes, homework sets, and solutions

Objective:

- Represent a sinusoid with a single complex number
- Express a sinusoid as seen on an oscilloscope as a complex number (a phasor)
- Determine the gain of a system from its oscilloscope traces



Phasor Voltages:

A generic sinusoid at frequency ω can be written as

$$x(t) = a \cos(\omega t) + b \sin(\omega t)$$

or

$$x(t) = r \cos(\omega t + \theta).$$

Note that to represent a sine wave, two terms are needed:

- The sine and cosine coefficients (termed rectangular form), or
- The amplitude (r) and phase shift (θ) (termed polar form).

Complex numbers can do that. The complex number representation for a sine wave is termed *its phasor representation*.

Euler's Identity

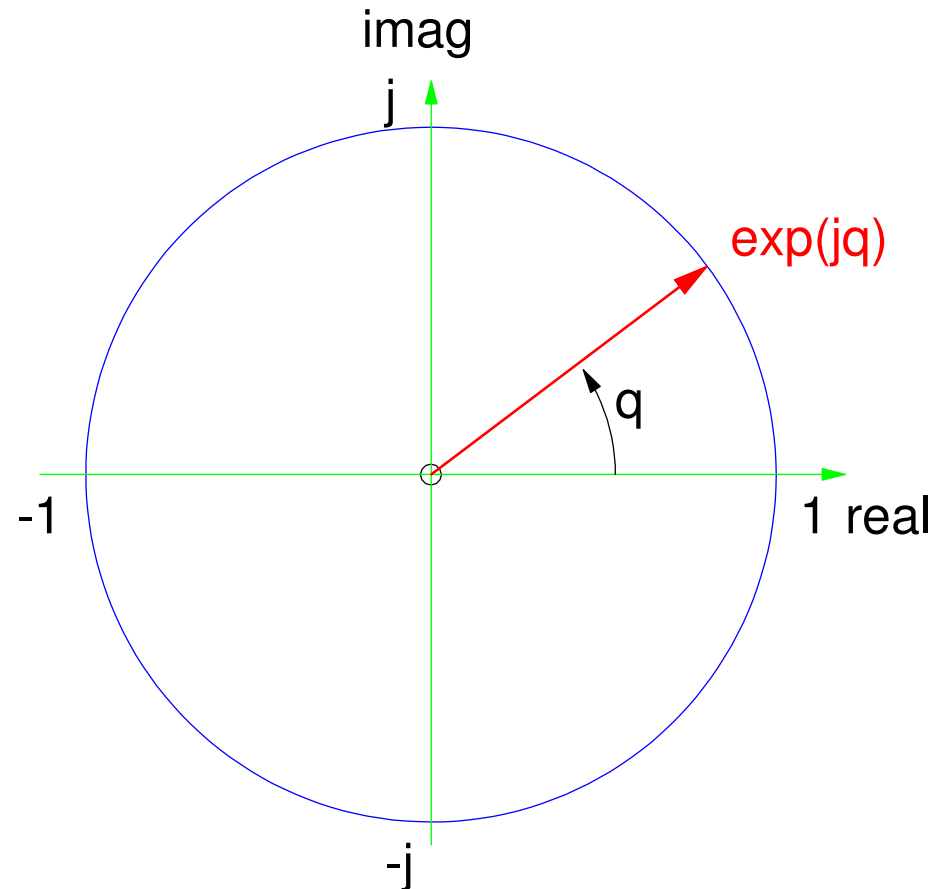
The heart of phasor representation is Euler's identity:

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

If you take the real part, you get cosine

$$\text{real}(e^{j\omega t}) = \cos(\omega t)$$

Hence, the phasor representation of cosine is 1.



Phasor Voltages: Rectangular Form

If you multiply by a complex number and take the real part, you get both sine and cosine:

$$(a + jb) \cdot e^{j\omega t} = (a + jb) \cdot (\cos(\omega t) + j \sin(\omega t))$$
$$= (a \cos(\omega t) - b \sin(\omega t)) + j(\dots)$$

$$\text{real}((a + jb) \cdot e^{j\omega t}) = a \cos(\omega t) - b \sin(\omega t)$$

$$a + jb \Leftrightarrow a \cos(\omega t) - b \sin(\omega t)$$

Phasor Voltages: Polar Form

Similarly, if you multiply a complex exponential with a complex number in polar form, you get a cosine with a phase shift:

$$\begin{aligned}(r\angle\theta) \cdot e^{j\omega t} &= (r \cdot e^{j\theta}) \cdot e^{j\omega t} \\&= r \cdot e^{j(\omega t + \theta)} \\&= r(\cos(\omega t + \theta) + j \sin(\omega t + \theta)) \\real((r\angle\theta) \cdot e^{j\omega t}) &= r \cdot \cos(\omega t + \theta)\end{aligned}$$

$r\angle\theta \Leftrightarrow r \cdot \cos(\omega t + \theta)$

Phasor Domain vs. Time Domain

Frequency is understood (not written) when using phasors

- Capital letters denote phasor-domain
- Lower case letters denote time-domain

Phasor Domain

$$V = 3 - j8$$

$$V = 8\angle -23^{\circ}$$

Time Domain

$$v(t) = 3 \cos(20t) + 8 \sin(20t)$$

$$v(t) = 8 \cos(20t - 23^{\circ})$$

Addition and Subtraction of Voltages

Phasor Domain

$$V_1 = 3 - j8$$

$$V_2 = 2 + j6$$

$$V_3 = V_1 + V_2$$

$$V_3 = 5 - j2$$

Time Domain

$$v_1 = 3 \cos(20t) + 8 \sin(20t)$$

$$v_2 = 2 \cos(20t) - 6 \sin(20t)$$

$$v_3 = v_1 + v_2$$

$$v_3 = 5 \cos(20t) + 2 \sin(20t)$$

Addition in Polar Form

This also works in polar form (use a calculator):

Phasor Domain

$$V_1 = 7\angle 15^\circ$$

$$V_2 = 9\angle 67^\circ$$

$$V_3 = V_1 + V_2$$

$$V_3 = 3.245 - j10.096$$

Time Domain

$$v_1 = 7 \cos (20t - 15^\circ)$$

$$v_2 = 9 \cos (20t + 67^\circ)$$

$$v_3 = v_1 + v_2$$

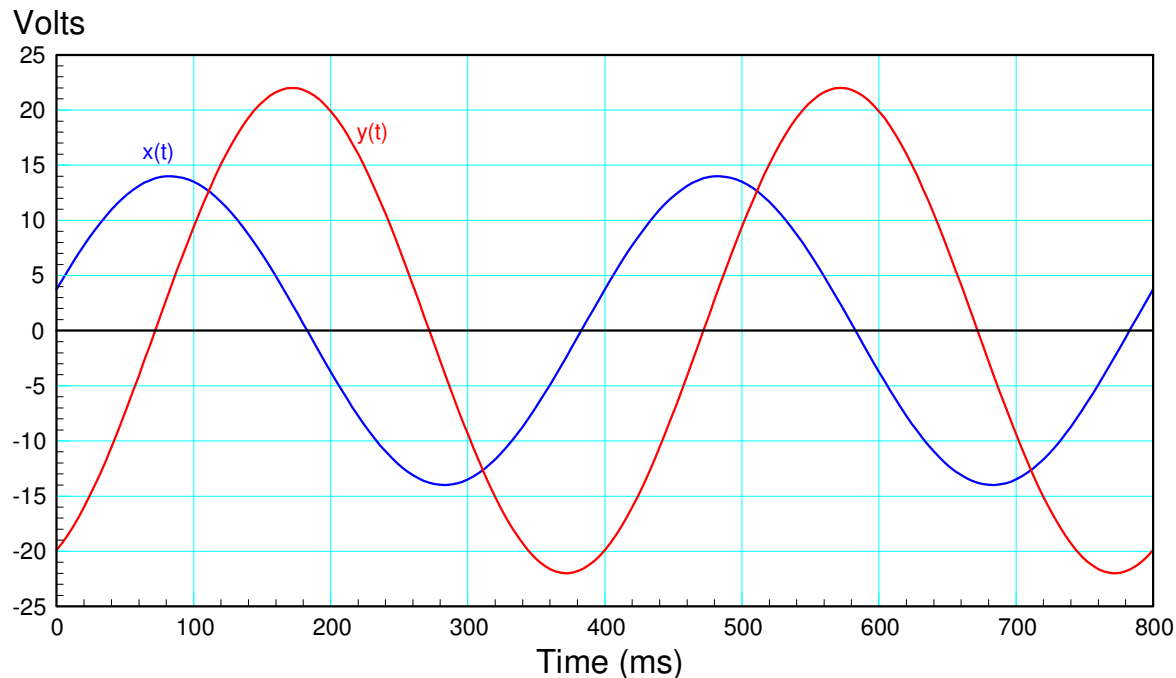
$$v_3 = 3.245 \cos (20t) + 10.096 \sin(20t)$$

note: V_3 is found using a calculator which does complex numbers

Phasor Voltages: Experimental

In lab, you normally express a voltage in polar form. For example, determine the following from the following signal from an oscilloscope:

- The frequency, and
- The phasor representation of X and Y



Frequency:

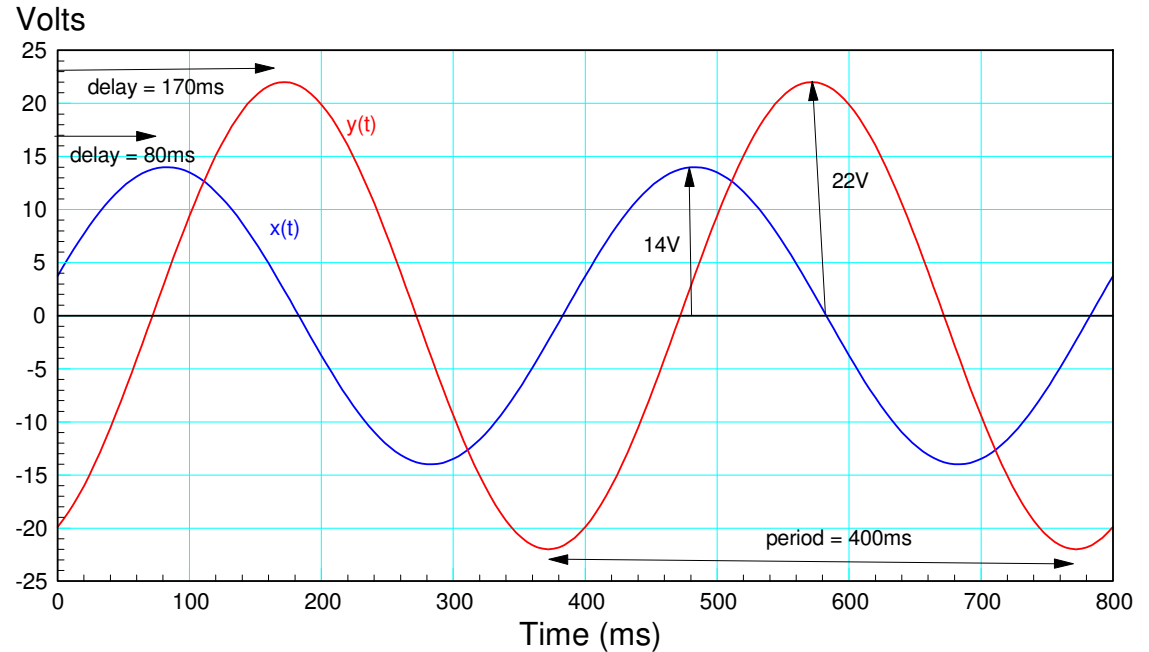
Frequency is defined as cycles per second or one over the period.

$$f = \frac{1}{T} \text{ Hz}$$

One cycle takes 400ms, so the frequency is

$$f = \frac{\text{one cycle}}{400\text{ms}} = 2.5\text{Hz}$$

$$\omega = 2\pi f = 5\pi \frac{\text{rad}}{\text{sec}}$$

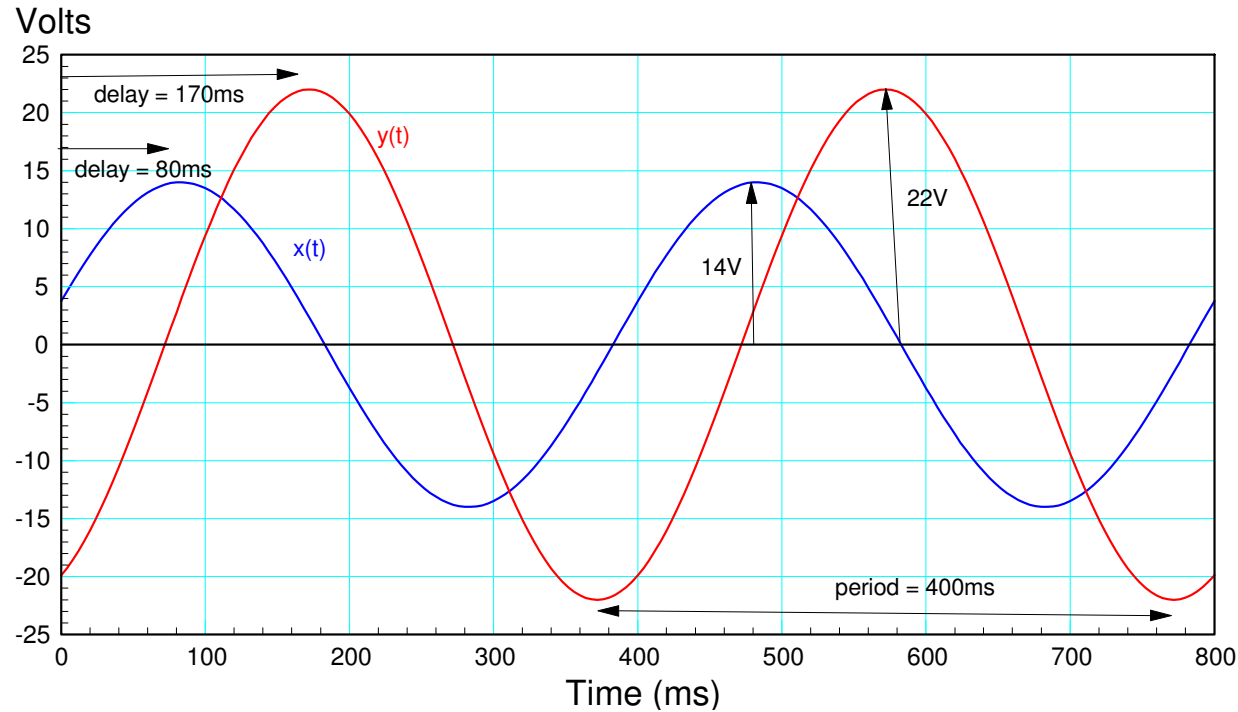


Peak Voltage:

The voltage from the average to the peak (V_p) is the amplitude:

$$|X| = 14V$$

$$|Y| = 22V$$



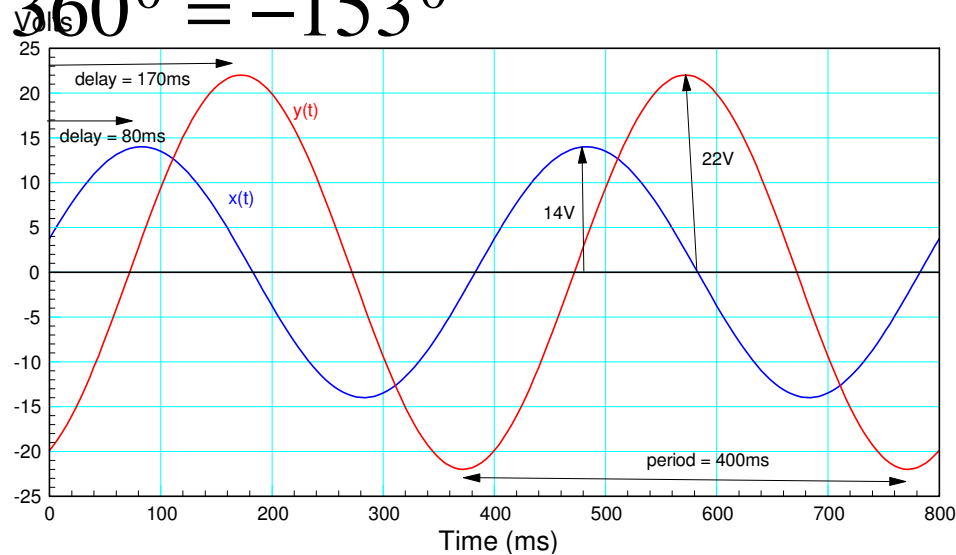
Phase Shift:

The delay is the phase shift (delay corresponds to a negative angle)

$$\theta = -\left(\frac{\text{delay}}{\text{period}}\right) \cdot 360^\circ$$

$$\theta_x = -\left(\frac{80\text{ms}}{400\text{ms}}\right) \cdot 360^\circ = -72^\circ$$

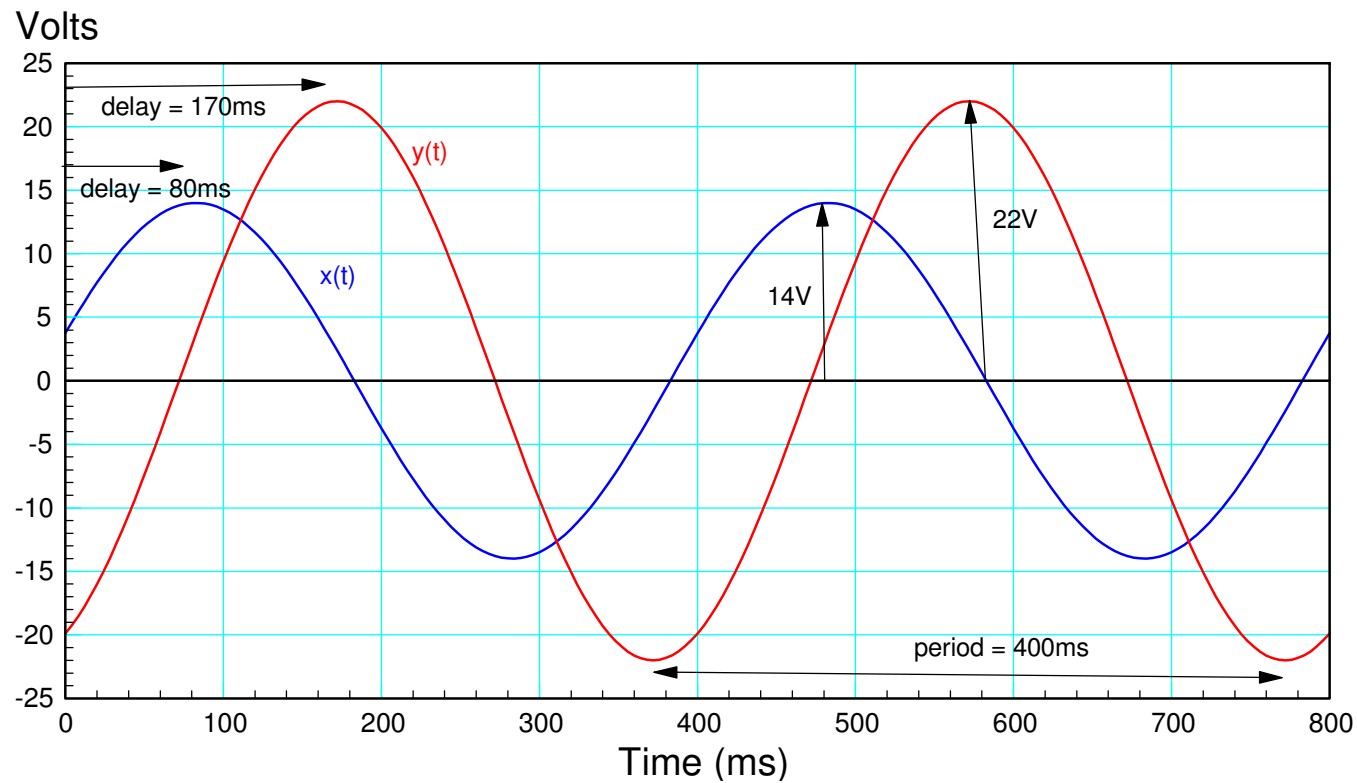
$$\theta_y = -\left(\frac{170\text{ms}}{400\text{ms}}\right) \cdot 360^\circ = -153^\circ$$



Result:

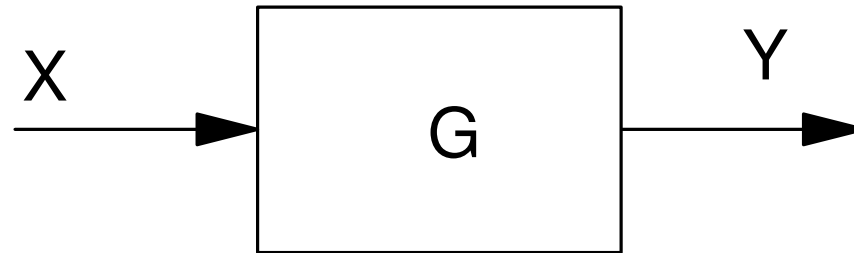
$$X = 14\angle -72^{\circ}$$

$$Y = 22\angle -153^{\circ}$$



Gain from X to Y:

A common problem with circuit analysis is to determine the gain of a circuit at a given frequency:



Gain is output / input

$$Y = G \cdot X$$

$$G = \frac{Y}{X}$$

Gain Computations:

- The amplitude is the ratio: $|Y| / |X|$
- The phase is the difference: $\theta_g = \theta_y - \theta_x$

Similarly, with the previous data

$$|G| = \frac{22V}{14V} = 1.571$$

$$\angle G = -\left(\frac{90\text{ms delay X to Y}}{400\text{ms period}}\right) \cdot 360^\circ = -81^\circ$$

The gain of this filter is

$$G = 1.571 \angle -81^\circ$$

