# Complex Power

## Objective:

- Define power for AC circuits
- Find the conditions for maximum power transfer
- Correct the power factor for a circuit

## AC Power:

For DC circuits, power is

$$P = VI$$

$$P = I^2R$$

$$P = \frac{V^2}{R}$$

For AC circuits, these still hold. The voltages and currents are time varying, however, so two types of power need to be defined:

*Instantaneous Power:* At any given moment in time, instantaneouns power is

$$P(t) = V(t) \cdot I(t)$$

*Average Power:* The average power over one cycle:

$$P_{avg} = \frac{1}{T}\int_T (V \cdot I)dt$$

Normally, we talk about the average power. This is also the RMS power:

$$P_{rms} = \frac{1}{2}V_{\max}I^*_{\max}$$

If you scale V and I by $\frac{1}{\sqrt{2}}$, you get RMS voltage and current. Then,

$$P_{rms} = V_{rms}I^*_{rms}$$

just like DC.

With AC signals, voltage and current might have a phase shift. If that phase shift is zero degrees, you should get a pure real power term. To get there, take the complex conjugate of current. (the '*' notation above means V times the complex conjugate of current).

Since

$$V = ZI$$

you have for AC circuits:

$$P_{rms} = V_{rms}I^*_{rms}$$

$$P_{rms} = (ZI_{rms})I^*_{rms}$$

$$P_{rms} = V_{rms}\left(\frac{V_{rms}}{Z}\right)^*$$

Note that

$$X \cdot X^* = |X|^2$$

So

$$P_{rms} = V_{rms} I^*_{rms}$$

$$P_{rms} = Z \cdot |I_{rms}|^2$$

$$P_{rms} = \frac{|V_{rms}|^2}{Z^*}$$

If you have resistors, inductors and capacitors in your circuit, the power will have real and complex terms.

- The real part of the power is energy absorbed. This is usually turned into heat or work for a motor.
- The complex part of the power is energy stored. If you have a +j term, the circuit looks like an inductor (termed inductive load). If you have a -j term, the circuit looks like a capacitor (termed capacitive load).

## Maximum Power to a Load:

Assume you can vary the load for a fixed source. What gives maximum power?

Assume the following circuit to model this:

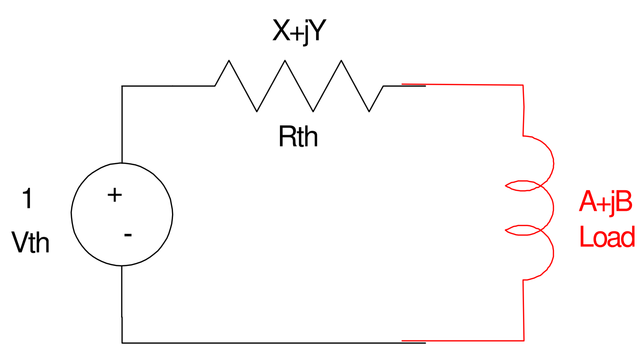


Figure 1:

The voltage at the load is from voltage division:

$$V_L = \left(\frac{A+jB}{(A+jB)+(X+jY)}\right)1$$

The current is

$$I_L = \frac{1}{(A+jB)+(X+jY)}$$

The power delivered is

$$P = V_L I_L^* = \left(\frac{A+jB}{(A+X)+j(B+Y)}\right)\left(\frac{1}{(A+X)-j(B+Y)}\right)$$

$$P = \left(\frac{A}{(A+X)^2+(B+Y)^2}\right) + j\left(\frac{B}{(A+X)^2+(B+Y)^2}\right)$$

The real part of the power is

$$P_{real} = \left(\frac{A}{(A+X)^2+(B+Y)^2}\right)$$

The maximum power is when the partial with respect to A and B are zero:

$$\frac{dP_{real}}{dB} = \left(\frac{-2(B+Y)}{\left((A+X)^2+(B+Y)^2\right)^2}\right) = 0$$

or

$$B = -Y$$

And

$$\frac{dP_{real}}{dA} = \left(\frac{(A+X)^2+(B+Y)^2-2A(A+X)}{\left((A+X)^2+(B+Y)^2\right)^2}\right) = 0$$

Since B+Y=0

$$((A+X)^2 - 2A(A+X)) = 0$$

$$(A+X)(A+X-2A) = 0$$

or

$$A = -X$$

$$A = X$$

The first requires negative resistance and produced energy - which is not what we want. The latter is the maximum power to the load:

**To deliver the maximum power to the load, make the load the complex conjugate of the source impedance.**

If you havea a capacitive source, you want an inductive load. If you have an inductive source, you want a capacitive load.

## Maximum Power to a Load (take 2):

Assume you can vary the source for a fixed load. What gives maximum power?

In this case, you can vary the source impedance (X+jY).

$$\frac{dP_{real}}{dY} = \left( \frac{-2(B+Y)}{\left((A+X)^2+(B+Y)^2\right)^2} \right) = 0$$

Y = -B

$$\frac{dP_{real}}{dX} = \left( \frac{-2A(A+X)}{\left((A+X)^2+(B+Y)^2\right)^2} \right) = 0$$

X = -A

Note that this asks you to have a source with a negative resistance. If that is not possible, you should make the source resistance as small as possible - zero if that's the best you can do:

**To maximize the power to the load, the source should have**

**- The complex part of its impedance should be equal and opposite to the load, and a**

**- The real part of its impedance should be as small as possible.**

Like you'd expect, a source with a smaller output impedance wastes less energy in it's own output impedance. It likewise delivers more power to the load.

## Power Factor Correction:

Can you reduce or eliminate the complex part of the load's inpedance?

Suppose Figure 1 represents a utility supplying power to a factory. The load is the factory, which has a real impedance (the energy they draw from the utility) and a complex part (due to the inductance of the motors in the factory, etc.) From the utility's standpoint, the complex part of the load costs them money. The complex impedance draws current from the power lines - which creates line losses in the source impedance. The customer doesn't pay for this power, however, since the corresponding complex power bounces back and forth from the customer to the utility 60 times per second.

Likewise, utilties have a term for the complex part of the power delivered to the customer: Volt Amp Reactive (VAR). Ideally, the utility would like each customer to consumer zero VARs since they don't pay for them but they cost the utliity money.

Power factor (pf) is a weighted measure of VAR:

$$pf = \left( \frac{\text{real power}}{\text{total power}} \right)$$

It is related to the angle of the impedance of the load:

$$Z_L = Z\angle\theta$$

$$pf = \cos(\theta)$$

Since cosine is an even function, you lose the information related to the sign of the angle. Sometimes, an additional term is added to include this information:

leading: $\theta > 0$ (inductive)

lagging: $\theta < 0$ (capacitive)

Ideally, all the power delivered to a customer is real, meaning the ideal power factor is one. To encourage customers to keep their loads close to real, a utility may fine a customer if his/her power factor drops below 0.95.

Problem: Suppose a customer has a load of

$$Z_L = 0.02 + j0.01$$

a) Find the customer's power factor.

b) Design a method to raise the power factor to 0.95.

a) The load is

$$Z_L = 0.02236\angle 26.56^0$$

The power factor is then

$$pf = \cos(26.56^0) = 0.8944$$

b) If you add a capacitor in parallel, it will add negative reactance, cancelling with the real reactance. Since it's in parallel, let's look at 1/Z:

$$\frac{1}{Z_{total}} = \frac{1}{Z_L} + j\omega C$$

$$\frac{1}{Z_{total}} = (44.72\angle -26.56^0) + (j\omega C)$$

$$\frac{1}{Z_{total}} = (40 - j20) + (j\omega C)$$

For the power factor to be 0.95:

$$\theta = a\cos(0.95) = 18.19^0$$

The real part of the admittance is 0.4. C doesn't affect that. The complex part needs to be

$$40 \cdot \tan(18.19^0) = \pm 13.15$$

So

$$-j20 + j\omega C = -j13.15$$

$$\omega C = 6.85$$

Assuming 60 Hz:

$$C = 0.018F$$

The company could place a large capacitor across its access to the utilities with a value of 0.018F. This is actually a common practice: the capacitors at the inlet to the plant provide short-term power for the inductive load in the factory (the complex part of the power). The utility provides the real part of the power (which is much more difficult to generate - it requires power plants, transmission lines, etc.)