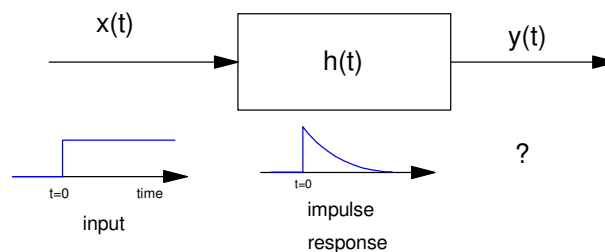


Convolution

So far, we know how to solve differential equations which have

- Sinusoidal inputs,
- A sum of sinusoidal inputs, and
- Periodic inputs.

Now let's look at how to solve differential equations when the input isn't periodic.



Background

Given a function, $y(t)$, its value at $t=0$ can be expressed in terms of a delta function as

$$\int_{-\infty}^{+\infty} y(t) \cdot \delta(t) \cdot dt = y(0)$$

Its value for all time can be evaluated as

$$y(t) = \int_{-\infty}^{\infty} y(\tau) \cdot \delta(t - \tau) \cdot d\tau$$

Suppose $h(t) = 1$, meaning you have a wire shorting the input to the output so that

$$y(t) = x(t)$$

$y(t)$ can be found using

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t - \tau) \cdot d\tau = x(t)$$

Now suppose $y(t)$ and $x(t)$ are related by a differential equation whose impulse response is $h(t)$. Now, $y(t)$ will come from

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t - \tau) \cdot d\tau$$

This is called *convolution*. The symbol for convolution is two stars:

$$y(t) = x(t) * h(t)$$

Graphical Interpretation of Convolution

Let's start with the differential equation we used before:

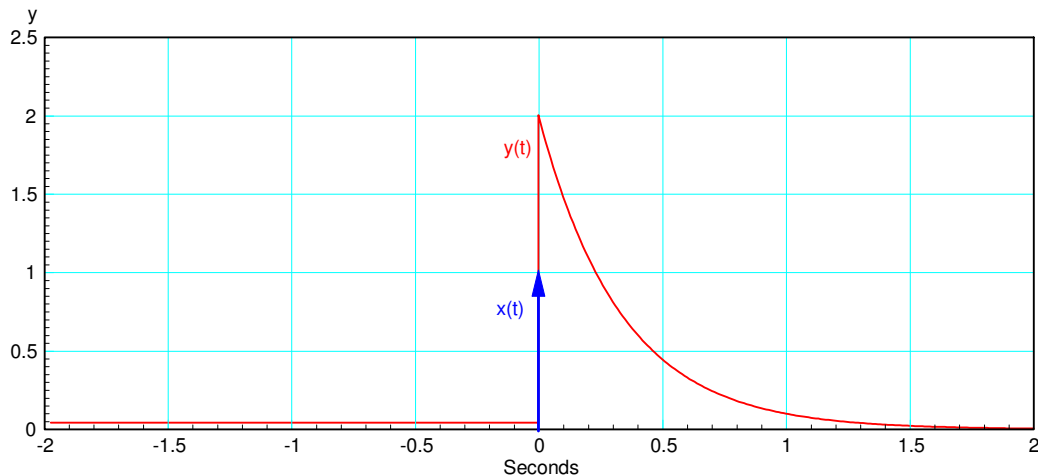
$$\frac{dy}{dt} + 3y = 2x$$

which has the impulse response

$$y(t) = 2e^{-3t} \cdot u(t)$$

Call this $h(t)$ (the impulse response)

$$h(t) = 2e^{-3t} \cdot u(t)$$



Impulse Response: $h(t)$

The generalized response to an input $x(t)$ is.

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t - \tau) \cdot d\tau$$

or

$$y(t) = \int_{-\infty}^{\infty} x(t - \tau) \cdot h(\tau) \cdot d\tau$$

(either way works)

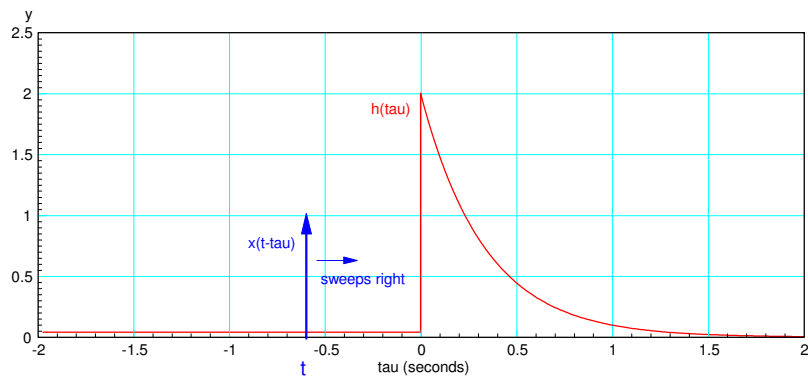
If $x(t)$ is an impulse

$$x(t) = \delta(t)$$

then, graphically, y at any given time, t , is found by

- Plotting the impulse response, $h(\tau)$ (red curve shown above),
- Plotting $x(t - \tau) = \delta(t - \tau)$, and
- Taking the area (integrating) of the product.

$y(t)$ for all time is found by sweeping $x(t - \tau)$ from $-\infty$ to $+\infty$.



The net result will give you $h(t)$, telling you that the impulse response of a filter is the impulse response.

$$y(t) = h(t) = 2e^{-3t}u(t)$$

Next, find $y(t)$ when $x(t)$ is a step input

$$x(t) = u(t)$$

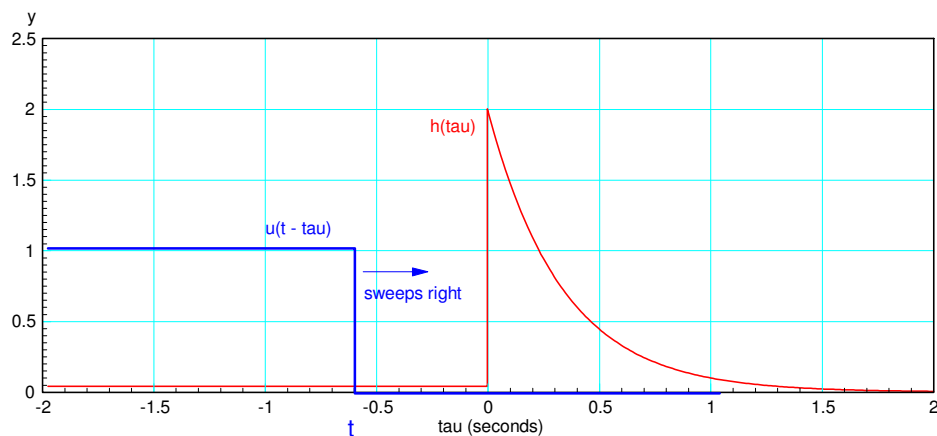
From before, $y(t)$ is from

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau) \cdot d\tau$$

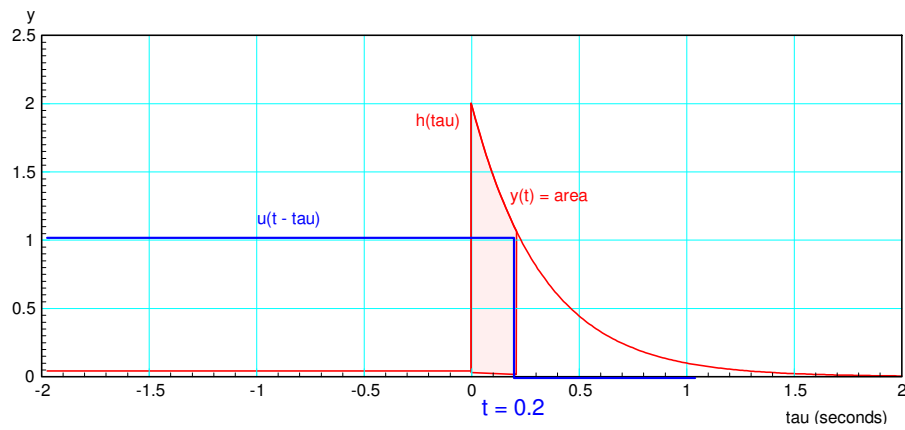
$$y(t) = \int_{-\infty}^{\infty} u(t-\tau) \cdot h(\tau) \cdot d\tau$$

Graphically,

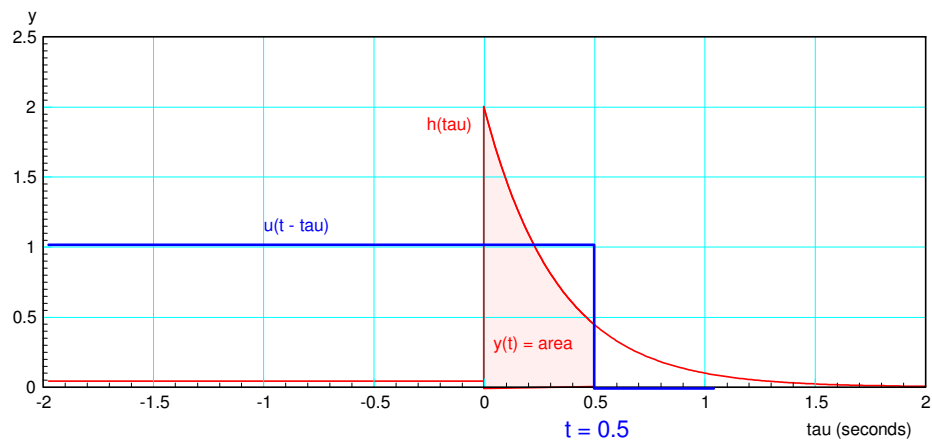
- $h(\tau)$ is the impulse response
- $u(t-\tau)$ is a flipped unit step that sweeps in from the left.



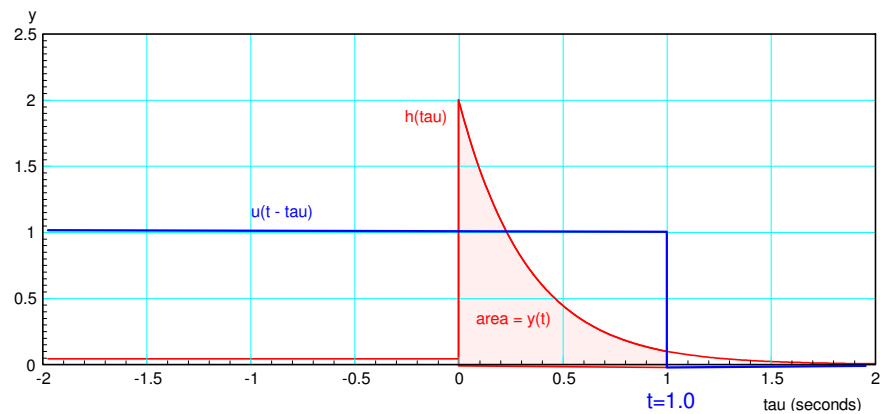
The area under the product of the two functions is $y(t)$. At $t = -0.6$, $y=0$ (no overlapping area)



At $t=0.2$, the common area ($y(t)$) increases



As time increases to $t = 0.5$, the common area increases further ($y(t)$ increases)



As time increases, almost all of the area of $h(t)$ is included.

This shows up with the mathematics

Computation of the Convolution Integral:

Find the step response of

$$\frac{dy}{dt} + 3y = 2x$$

for

$$x(t) = u(t)$$

Solution using Convolution: First, find the impulse response

$$h(t) = 2e^{-3t}u(t)$$

Next, evaluate the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau) \cdot d\tau$$

$$y(t) = \int_{-\infty}^{\infty} u(t-\tau) \cdot 2e^{-3\tau}u(\tau) \cdot d\tau$$

Note that

- $u(\tau) = 0$ for $\tau < 0$

This lets you reset the lower bound on the integral to zero

- $u(t-\tau) = 0$ for $\tau > t$

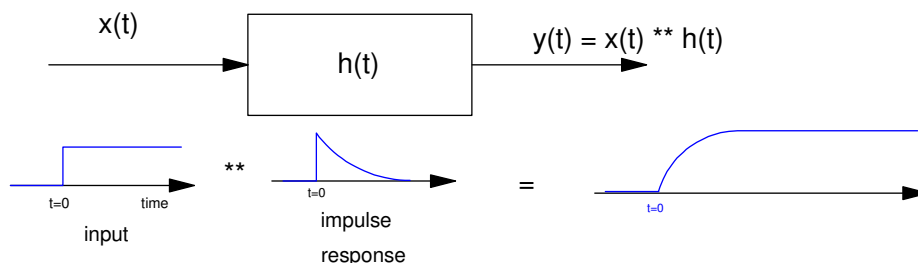
This lets you reset the upper bound on the integral to t

$$y(t) = \int_0^t 2e^{-3\tau} \cdot d\tau$$

$$y(t) = \left(\frac{-2}{3} e^{-3\tau} \right)_0^t$$

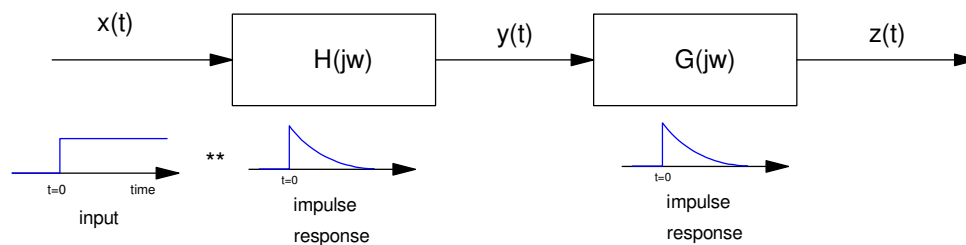
Also note that $y(t) = 0$ for $t < 0$ (this is a causal system: the output won't happen before the input)

$$y(t) = \left(\frac{2}{3} - \frac{2}{3} e^{-3t} \right) u(t)$$



The output is the convolution of $x(t)$ and $h(t)$: $y(t) = x(t) ** h(t)$

Convolution (take 2): Suppose you ran the output through another filter:



Find the output of Cascaded Filters

The same result holds: the output of each filter is the convolution of its input with the filter's impulse response

- $y(t) = x(t) ** h(t)$
- $z(t) = y(t) ** g(t)$

If the impulse response of $G(j\omega)$ is

$$g(t) = 4e^{-t}u(t)$$

then

$$z(t) = (4e^{-5t}u(t)) * \left(\left(\frac{2}{3} - \frac{2}{3}e^{-3t} \right) u(t) \right)$$

$$z(t) = \int_{-\infty}^{\infty} (4e^{-5(t-\tau)}u(t-\tau)) \cdot \left(\left(\frac{2}{3} - \frac{2}{3}e^{-3\tau} \right) u(\tau) \right) \cdot d\tau$$

It's doable - but there has to be a better way...