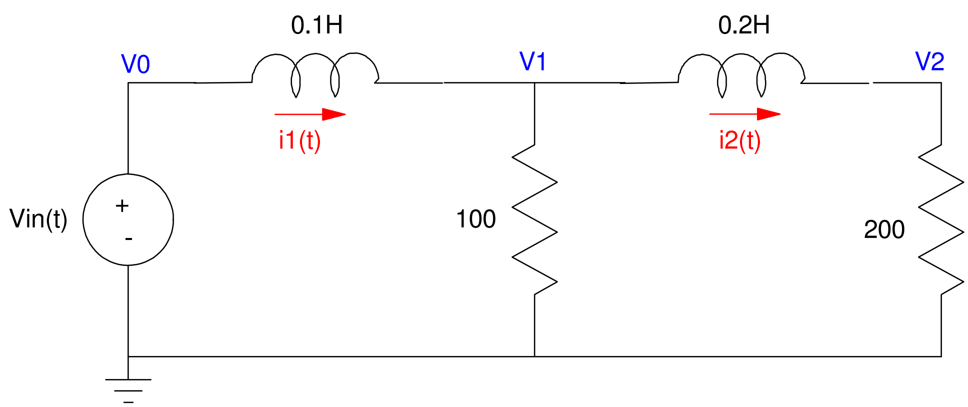
Voltage Nodes in the LaPlace Domain

Note: The parallel model for inductors and capacitors work better when writing voltage node equations.

Example 1: Find V2(t) for the following circuit. Assume

$$v_{in}(t) = \begin{cases} 5V & t < 0 \\ 0V & t > 0 \end{cases}$$

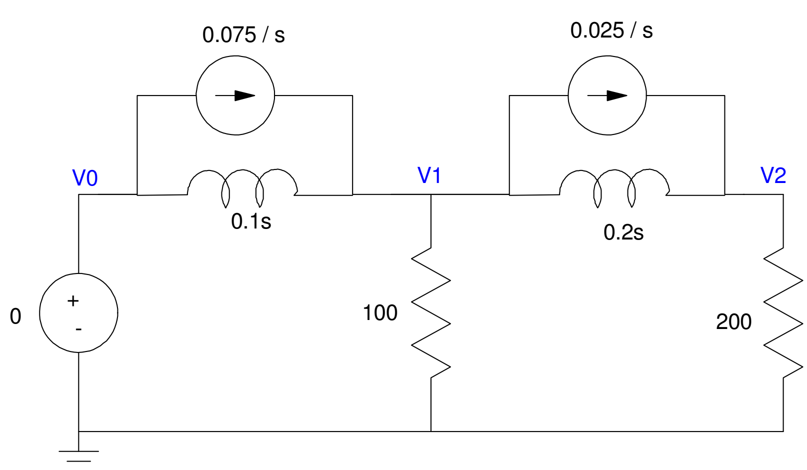


From before, the current at t=0 is

$$i_1(0) = 75mA$$

$$i_2(0) = 25mA$$

The parallel model (for voltage nodes) would be



with the voltage node equations being

$$\left(\frac{V_1}{0.1s}\right)-\left(\frac{0.075}{s}\right)+\left(\frac{V_1}{100}\right)+\left(\frac{0.025}{s}\right)+\left(\frac{V_1-V_2}{0.2s}\right)=0$$

$$\left(\frac{V_2-V_1}{0.2s}\right)+\left(\frac{V_2}{200}\right)-\left(\frac{0.025}{s}\right)=0$$

Grouping terms to solve for V2(s)

$$\left(\frac{1}{0.1s}+\frac{1}{100}+\frac{1}{0.2s}\right)V_1-\left(\frac{1}{0.2s}\right)V_2=\left(\frac{0.05}{s}\right) \qquad (1)$$

$$-\left(\frac{1}{0.2s}\right)V_1+\left(\frac{1}{0.2s}+\frac{1}{200}\right)V_2=\left(\frac{0.025}{s}\right) \qquad (2)$$

Solve: Method #1: Gauss elimination

$$aV_1-bV_2=c \qquad * \text{ b}$$

$$-bV_1+dV_2=e \qquad * \text{ a}$$

$$(-b^2+ad)V_2=bc+ae$$

$$V_2=\left(\frac{bc+ae}{ad-b^2}\right)$$

Plugging back in the values of (a, b, c, d, e)

$$V_2=\left(\frac{\left(\frac{1}{0.2s}\right)\left(\frac{0.05}{s}\right)+\left(\frac{1}{0.1s}+\frac{1}{100}+\frac{1}{0.2s}\right)\left(\frac{0.025}{s}\right)}{\left(\frac{1}{0.1s}+\frac{1}{100}+\frac{1}{0.2s}\right)\left(\frac{1}{0.2s}+\frac{1}{200}\right)-\left(\frac{1}{0.2s}\right)^2}\right)$$

Simplify (30 minutes later....)

$$V_2=\left(\frac{5(s+2500)}{(s+500)(s+2000)}\right)$$

Method #2: State Space. Place the system in the form of

$$sX=AX+BU$$

$$Y=CX+DU$$

Rewrite equations (1) and (2)

$$(0.01s+15)V_1-(5)V_2=(0.05) \qquad (1)$$

$$-(5)V_1+(0.005s+5)V_2=(0.025) \qquad (2)$$

Solve for the highest derivative

$$sV_1 = -1500V_1 + 500V_2 + 5 \quad (1)$$

$$sV_2 = 1000V_1 - 1000V_2 + 5 \quad (2)$$

Place in matrix form (note: U(s) = 1, meaning the input is a delta function for this circuit)

$$\begin{bmatrix} sV_1 \\ sV_2 \end{bmatrix} = \begin{bmatrix} -1500 & 500 \\ 1000 & -1000 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$Y = V_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + [0]$$

Place in Matlab

```
A = [-1500,500;1000,-1000]
B = [5;5]
C = [0,1]
D = 0;
G = ss(A,B,C,D);
zpk(G)

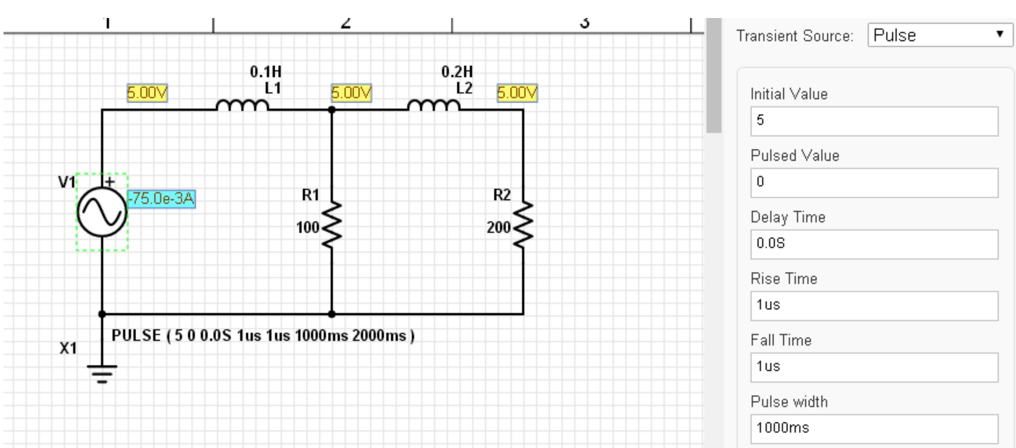
           5 (s+2500)
V2(s) = ----------------
        (s+2000) (s+500)
```

To find v2(t), take the inverse-LaPlace transform

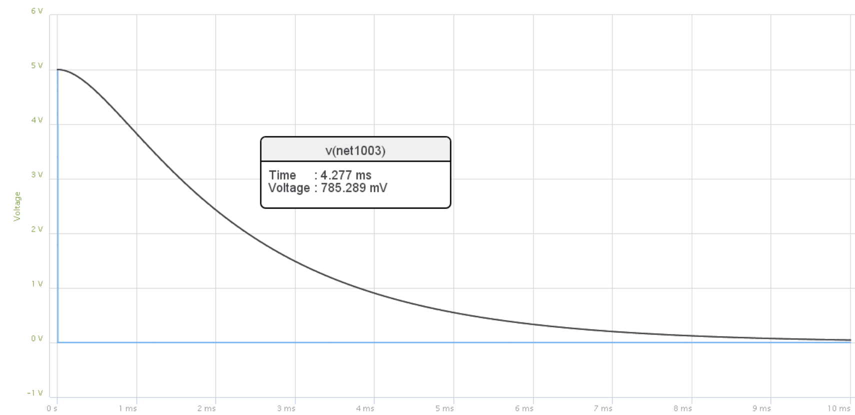
$$V_2 = \left(\frac{5(s+2500)}{(s+500)(s+2000)}\right) = \left(\frac{6.667}{s+500}\right) + \left(\frac{-1.667}{s+2000}\right)$$

$$v_2(t) = (6.667e^{-500t} - 1.667e^{-2000t})u(t)$$

Checking in PartSim: Input the circuit with the input being a pulse source



Run a transient simulation:



This matches up with the Matlab solution

```
t = [0:0.0001:0.01]';
y = impulse(G, t);
plot(t,y);
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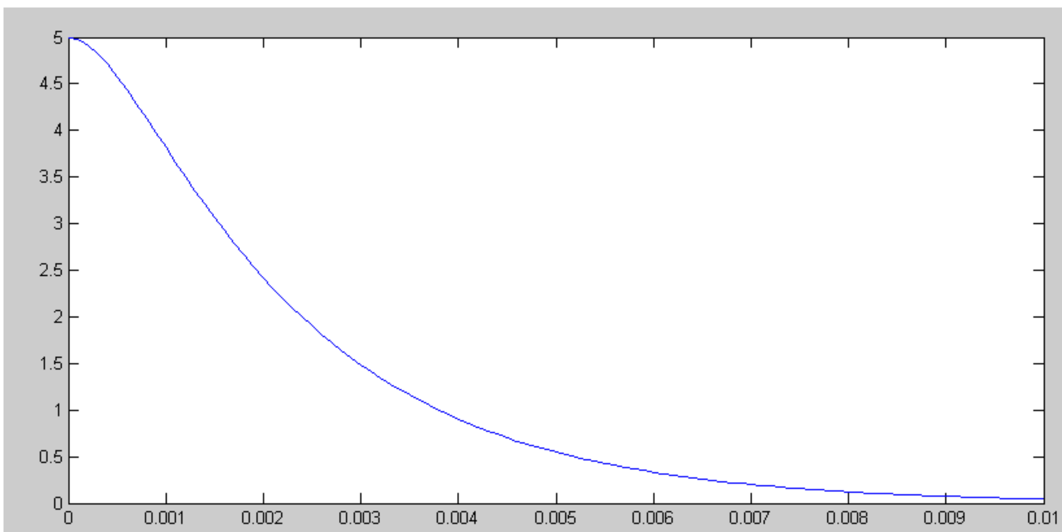
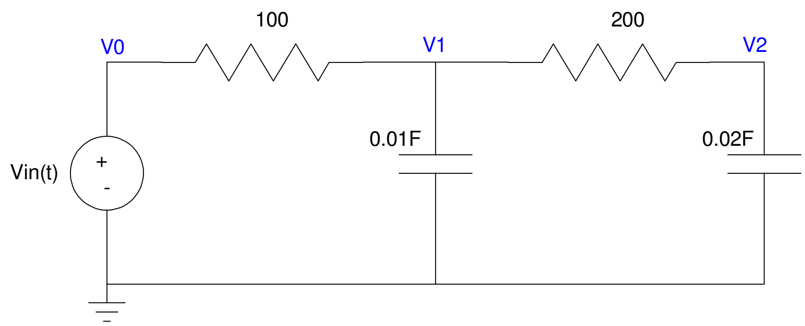


Voltage Nodes with Capacitors

Example 2: Find v2(t) assuming

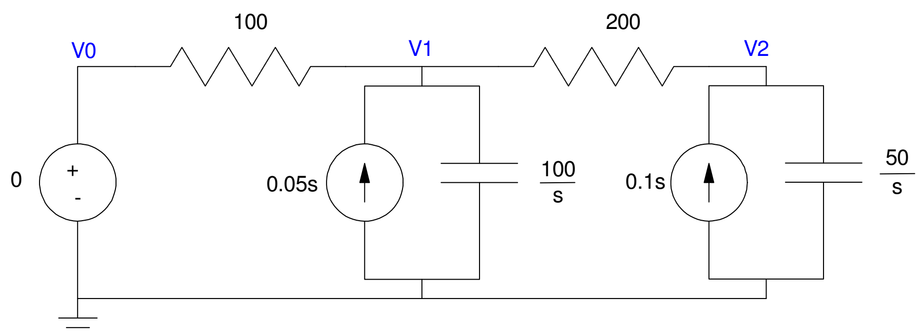
$$v_{in}(t) = \begin{cases} 5V & t < 0 \\ 0V & t > 0 \end{cases}$$



Solution: Find the initial conditions. Capacitors are open circuits at DC, resulting in

$$v_1(0) = v_2(0) = 5V$$

Convert to LaPlace using the parallel model



Write the voltage node equations

$$\left(\frac{V_1}{100}\right) - 0.05s + \left(\frac{V_1}{100/s}\right) + \left(\frac{V_1 - V_2}{200}\right) = 0 \qquad (1)$$

$$\left(\frac{V_2 - V_1}{200}\right) - 0.01s + \left(\frac{V_2}{50/s}\right) = 0 \qquad (2)$$

Simplify and group terms

$$(0.01s+0.015)V_1-(0.005)V_2=0.05s$$
$$(-0.005)V_1+(0.02s+0.005)V_2=0.1s$$

Solve: Method #1: Gauss Elimination. Express this as

$$aV_1-bV_2=c \qquad * \text{ b}$$
$$-bV_1+dV_2=e \qquad * \text{ a}$$

add the two equations

$$(-b^2+ad)V_2=bc+ae$$
$$V_2=\left(\frac{bc+ae}{ad-b^2}\right)$$

Substituting in for (a, b, c, d, e)

$$V_2=\left(\frac{(0.005)(0.05s)+(0.01s+0.015)(0.1s)}{(0.01s+0.015)(0.02s+0.005)-(0.005)^2}\right)$$

Method #2: State Space. Rewrite (1) and (2) as

$$sV_1=-15V_1+0.5V_2+5s$$
$$sV_2=0.25V_1-0.25V_2+5s$$

Place in matrix form

$$\begin{bmatrix} sV_1 \\ sV_2 \end{bmatrix}=\begin{bmatrix} -15 & 0.5 \\ 0.25 & -0.25 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}+\begin{bmatrix} 5 \\ 5 \end{bmatrix}s$$
$$Y=V_2=\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}+[0]$$

Place in Matlab

```
A = [-15,0.5 ; 0.25,-0.25];
B = [5;5];
C = [0,1];
D = 0;
G = ss(A,B,C,D);
zpk(G)

        5 (s+15.25)
G = --------------------
    (s+15.01) (s+0.2415)
```

This is the impulse response (U(s) = 1). We want U(s) = s, so multiply by 's'

```
        5s(s+15.25)
V2 = --------------------
    (s+15.01) (s+0.2415)
```

Capacitors

From before, the VI characteristics for a capacitor is

$$i(t) = C\frac{dv}{dt}$$

Taking the LaPlace transform

$$I(s) = C \cdot (sV - v(0))$$

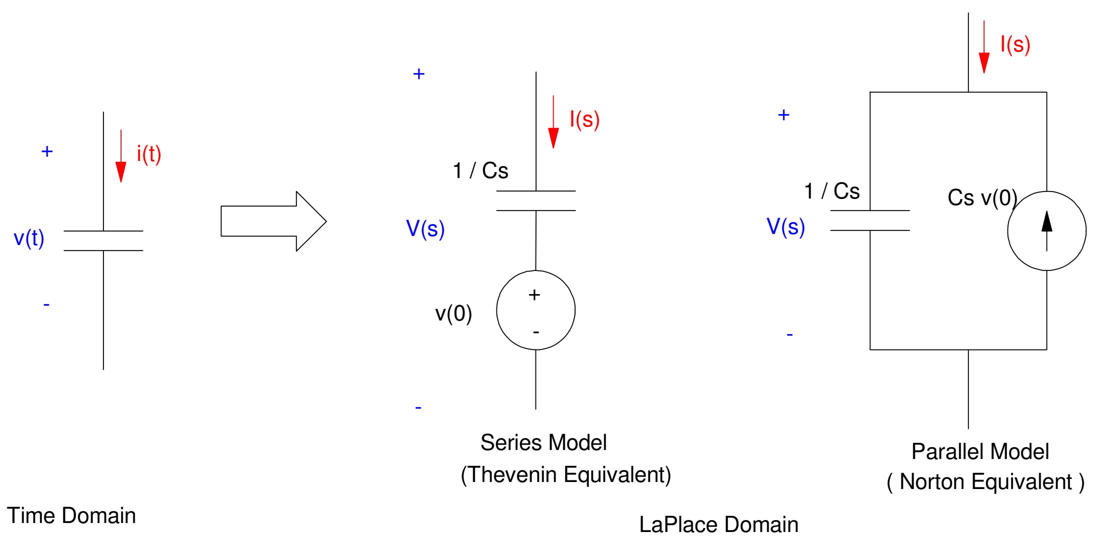
$$I(s) = CsV - Cv(0)$$

Solving for V

$$V = \left(\frac{1}{Cs}\right) I + v(0)$$

This gives the series (Thevenin) model for a capacitor. The parallel model has

$$I_{short} = \frac{V_{th}}{R_{th}} = \frac{v(0)}{1/Cs} = Csv(0)$$

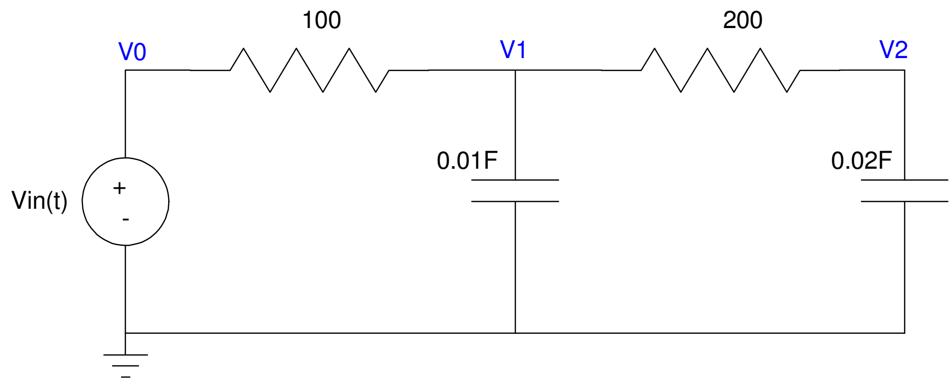


Note that

- The series model is more useful when writing current loop equations
- The parallel model is more useful when writing votlage node equations.

Example: Convert the following circuit to the LaPlace domain and write the voltage node and current loop equations.

$$v_{in}(t) = \begin{cases} 5V & t < 0 \\ 10V & t > 0 \end{cases}$$



Solution: First find the initial conditions. For t<0, the voltage is a constant (5V). Using phasor analysis,

$$C \rightarrow \frac{1}{j\omega C} = \infty$$

$$V_1 = V_2 = 5V$$

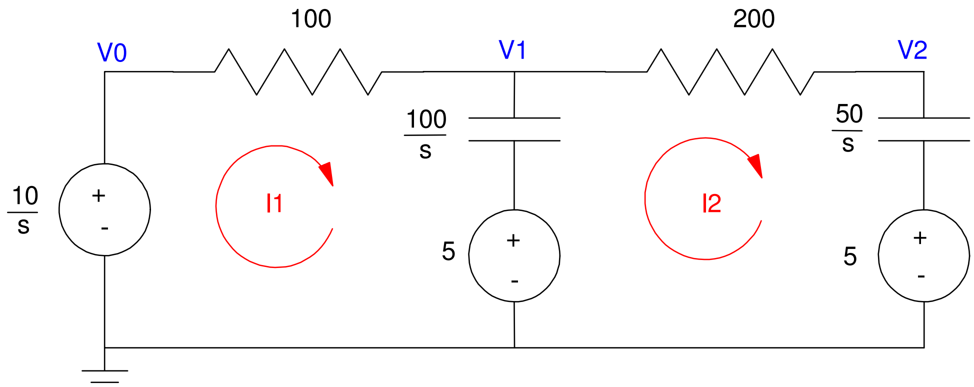
For the series (Thevenin) model, you have a voltage source with

$$V_{th} = v(0) = 5$$

For the parallel (Norton) model, you have a current source with

$$I_N = \frac{V_{th}}{R_{th}} = \frac{v(0)}{1/Cs} = C \cdot v(0) \cdot s$$

Series Model for t > 0:

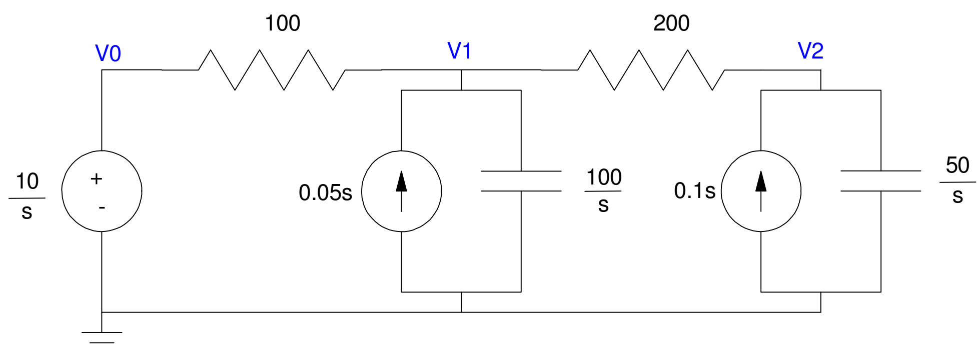


The current loop equations are then:

$$-\frac{10}{s} + 100I_1 + \left(\frac{100}{s}\right)(I_1 - I_2) + 5 = 0$$

$$-5 + \left(\frac{100}{s}\right)(I_2 - I_1) + 200I_2 + \left(\frac{50}{s}\right)I_2 + 5 = 0$$

Parallel Model for t > 0:



the voltage node equations become

$$V_0 = \frac{10}{s}$$

$$\left(\frac{V_1 - V_0}{100}\right) - 0.05s + \left(\frac{V_1}{100/s}\right) + \left(\frac{V_1 - V_2}{200}\right) = 0$$

$$\left(\frac{V_2 - V_1}{200}\right) - 0.1s + \left(\frac{V_2}{50/s}\right) = 0$$