State Variables in the LaPlace Domain

There's actually an easier way to write the equations for RLC circuits. Express the system in state-space form

$$sX = AX + BU$$

$$Y = CX + DU$$

Let the states be the terms that define the energy in the system

- currents for inductors
- voltages for capacitors

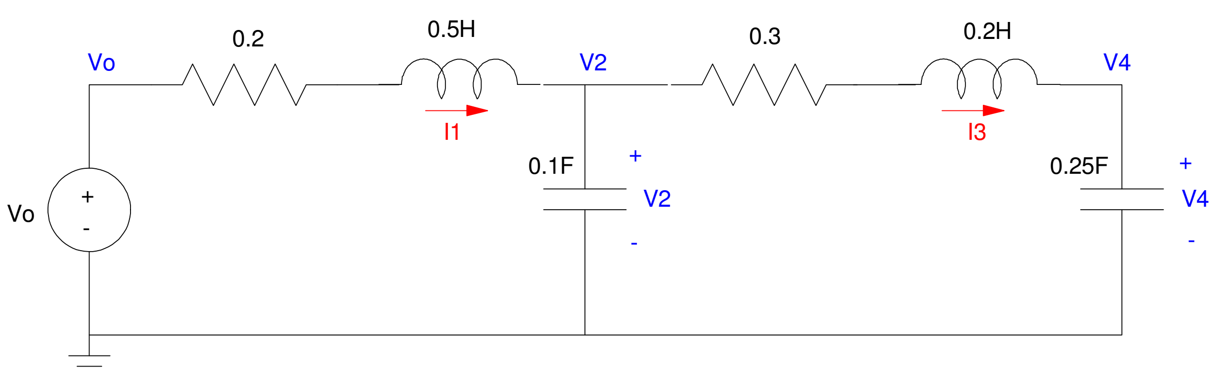
If you have N energy storage elements, you have N states. To determine (A, B), determine the

- current to capacitors $\left(i = C\frac{dv}{dt}\right)$
- voltages across inductors $\left(v = L\frac{di}{dt}\right)$

in terms of the states.

Example 1: Determine v4(t) for the following circuit assuming

$$v_{in}(t) = \begin{cases} 5V & t < 0 \\ 0V & t > 0 \end{cases}$$



Step 1: Determine the initial conditions. Capacitors are open circuits at DC and inductors are shorts at DC. This results in

$$i_1(0) = i_3(0) = 0A$$

$$v_2(0) = v_4(0) = 5V$$

Step 2: There are 4 energy storage elements (capacitors and inductors). There are 4 state variables. Define these to be

$$X = \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}$$

Write 4 coupled differential equations for this circuit

$$v_1 = L_1 \frac{di_1}{dt} = V_0 - 0.2I_1 - V_2$$

$$i_2 = C_2 \frac{dv_2}{dt} = I_1 - I_3$$

$$v_3 = L_3 \frac{di_3}{dt} = V_2 - 0.3I_3 - V_4$$

$$i_4 = C_4 \frac{dv_4}{dt} = I_3$$

Take the LaPlace transform

$$L_1(sI_1 - i_1(0)) = V_0 - 0.2I_1 - V_2$$

$$C_2(sV_2 - v_2(0)) = I_1 - I_3$$

$$L_3(sI_3 - i_3(0)) = V_2 - 0.3I_3 - V_4$$

$$C_4(sV_4 - v_4(0)) = I_3$$

Solve for the highest derivative

$$sI_1 = 2V_0 - 0.4I_1 - 2V_2 + i_1(0)$$

$$sV_2 = 10I_1 - 10I_3 + v_2(0)$$

$$sI_3 = 5V_2 - 1.5I_3 - 5V_4 + i_3(0)$$

$$sV_4 = 4I_3 + v_4(0)$$

Place in matrix form

$$\begin{bmatrix} sI_1 \\ sV_2 \\ sI_3 \\ sV_4 \end{bmatrix} = \begin{bmatrix} -0.4 & -2 & 0 & 0 \\ 10 & 0 & -10 & 0 \\ 0 & 5 & -1.5 & -5 \\ 0 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} i_1(0) \\ v_2(0) \\ i_3(0) \\ v_4(0) \end{bmatrix}$$

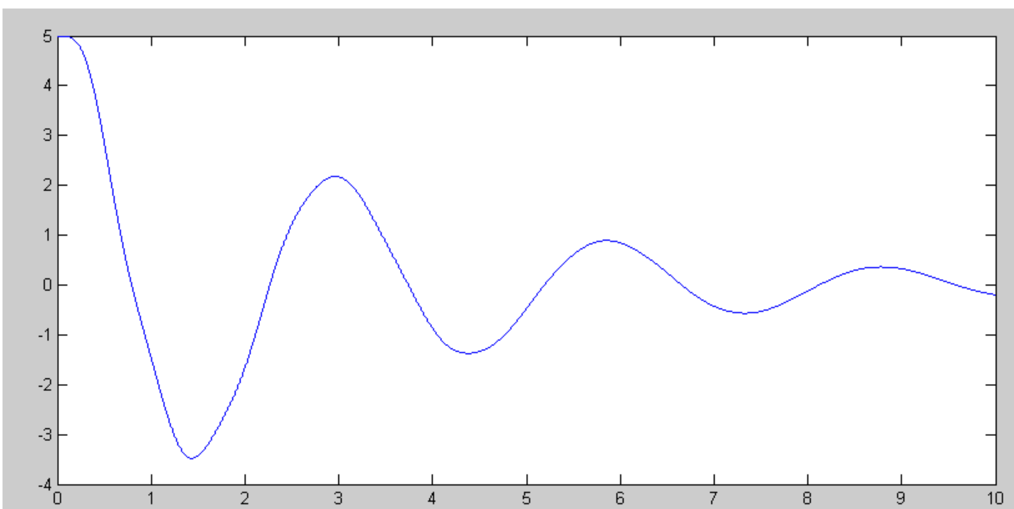
$$Y = V_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix} + [0]$$

Place in Matlab. Note that the B matrix is just the initial conditions.

```
A = [-0.4,-2,0,0 ; 10,0,-10,0 ; 0,5,-1.5,-5 ; 0,0,4,0 ]

   -0.4000   -2.0000         0         0
   10.0000         0  -10.0000         0
         0    5.0000   -1.5000   -5.0000
         0         0    4.0000         0

B = [0 ; 5 ; 0 ; 5]

     0
     5
     0
     5

C = [0,0,0,1];
D = 0;
G = ss(A,B,C,D);
zpk(G)

              5 (s+0.646) (s^2 + 1.254s + 89.79)
V4(s) = ------------------------------------------
         (s^2 + 0.6102s + 4.7) (s^2 + 1.29s + 85.11)
```

The inverse LaPlace transform is then

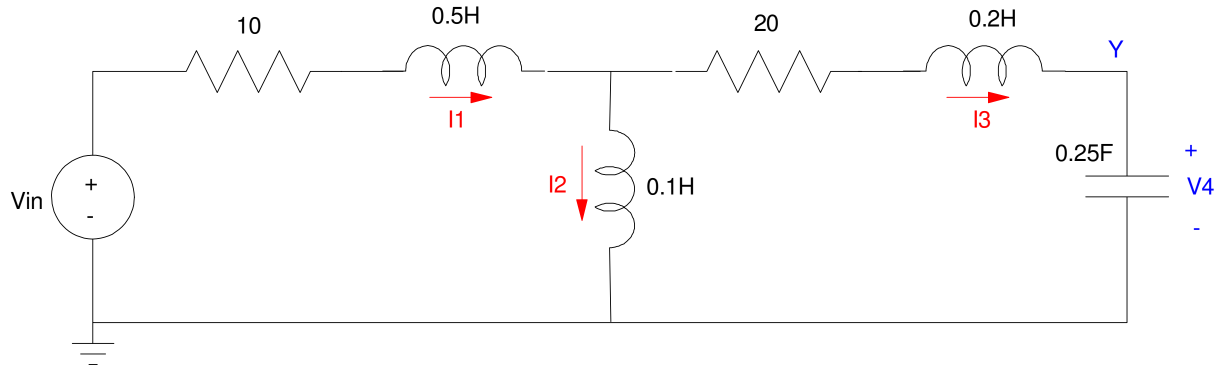
```
t = [0:0.0001:10]';
y = impulse(Y,t);
plot(t,y);
```



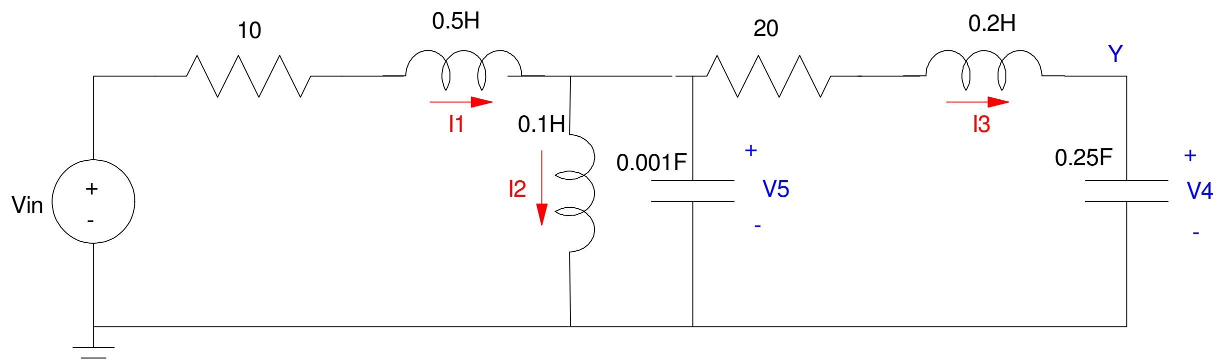
Example 2: Sometimes the circuit is difficult to solve. In that case do a typical engineering solution:

- Change the problem so that it's solvable (or easier to solve), but
- Make sure the changes don't change the flavor of the problem

Find V4(t):



This problem is actually hard due to the constraint that I1 = I2 + I3. The problem is there isn't a capacitor at the middle node. So... add a capacitor there. Just make it small so it doesn't change much



Now write 5 coupled differential equations

$$v_1 = L_1 \frac{di_1}{dt} = V_{in} - 10I_1 - V_5$$

$$v_2 = L_2 \frac{di_2}{dt} = V_5$$

$$v_3 = L_3 \frac{di_3}{dt} = V_5 - 20I_3 - V_4$$

$$i_4 = C_4 \frac{dv_4}{dt} = I_3$$

$$i_5 = C_5 \frac{dv_5}{dt} = I_1 - I_2 - I_3$$

Take the LaPlace transform (note that Vin(t) = 0)

$$sI_1 = -20I_1 - 2V_5 + i_1(0)$$

$$sI_2 = 10V_5 + i_2(0)$$

$$sI_3 = 5V_5 - 100I_3 - 5V_4 + i_3(0)$$

$$sV_4 = 4I_3 + v_4(0)$$

$$sV_5 = 1000I_1 - 1000I_2 - 1000I_3 + v_5(0)$$

Place in matrix form.

$$\begin{bmatrix} sI_1 \\ sI_2 \\ sI_3 \\ sV_4 \\ sV_5 \end{bmatrix} = \begin{bmatrix} -10 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & -100 & -5 & 5 \\ 0 & 0 & 4 & 0 & 0 \\ 1000 & -1000 & -1000 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_4 \\ V_5 \end{bmatrix} + \begin{bmatrix} i_1(0) \\ i_2(0) \\ i_3(0) \\ v_4(0) \\ v_5(0) \end{bmatrix}$$

$$Y = V_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_4 \\ V_5 \end{bmatrix} + [0]$$

Solve in Matlab. Note that the initial condition is

$$\begin{bmatrix} i_1(0) \\ i_2(0) \\ i_3(0) \\ v_4(0) \\ v_5(0) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
A = [-10,0,0,0,-2 ; 0,0,0,0,10 ; 0,0,-100,-5,5 ; 0,0,4,0,0 ;
1000,-1000,-1000,0,0]

A =

         -10            0            0            0           -2
           0            0            0            0           10
           0            0         -100           -5            5
           0            0            4            0            0
        1000        -1000        -1000            0            0

B = [0.5;0;0;0;0]

B =
```

```
    0.5000
         0
         0
         0
         0

C = [0,0,0,1,0];
D = 0;
G = ss(A,B,C,D);
zpk(G)
```

```
                                     10000 s
Y(s) = ----------------------------------------------------------
        (s+78.48) (s+8.277) (s+0.2006) (s^2 + 23.04s + 1.535e004)
```

The time response is

```
t = [0:0.001:5]';
y = impulse(G, t);
plot(t,y);
```