Superposition

Linear Systems

Linear systems have the property:

f(a+b) = f(a) + f(b)

A large class of circuits are linear. These are described by ordinary differential equations such as

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x$$

Resistors, capacitors, and inductors are linear devices and produce linear differential equations of this form. An example of a function which is *not* linear is a threshold function (like a diode)

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x > 1 \end{cases}$$

In this case

$$f(0.6 + 0.7) \neq f(0.6) + f(0.7)$$

$$1 \neq 0 + 0$$

An example of a differential equation which is nonlinear is

$$\frac{d^2 y}{dt^2} + \left(\frac{dy}{dt}\right)^2 + y \cdot \frac{dy}{dt} = x$$

As a rule of thumb, as long as you don't have diodes or other nonlinear devices in your circuit, it will behave as a linear system.

If you have a linear system, you can split a complex problem into several simpler problems. This is the idea behind superposition. For example, suppose you have the forcing function

$$x(t) = 2 + 3\cos(4t) + 5\cos(6t).$$

If y(t) is a function of x(t):

$$y = f(x)$$

then

$$y = f(2 + 3\cos(4t) + 5\cos(6t))$$

If the system is linear, this is equivalent to

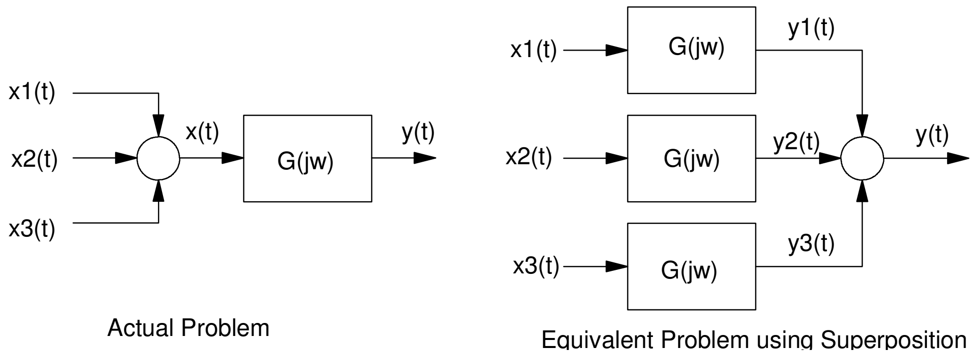
$$y = f(2) + f(3\cos(4t)) + f(5\cos(6t))$$

To find y(t)

- Treat this as three separate problems.
- Find y(t) for each input, ignoring the other inputs
- The total output is then the sum of each of these separate problems.

Pictorially, this looks like the following:

- To find the output of a filter with three separate inputs,
- Treat this as three copies of that filter, each operating on a separate input.
- Sum the result to get the total output



Using Superposition, you can treat a problem with multiple inputs as multiple problems, each with a single input.

For example, find the solution to the following differential equation:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 20x$$

when

$$x(t) = 2 + 3\cos(4t) + 5\sin(6t)$$

Solution: Treat this as three separate problems

$$x_1(t) = 2$$

$$x_2(t) = 3\cos(4t)$$

$$x_3(t) = 5\cos(6t)$$

To solve this differential equation for a sinusoidal input, convert to phasor notation. The differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 20x$$

becomes

$$(j\omega)^2 Y + 2(j\omega)Y + 10Y = 20X$$

$$Y = \left(\frac{20}{(j\omega)^2 + 2j\omega + 10}\right)X$$

Now, solve each problem separately.

$$x_1(t) = 2$$

In phasor form

$$X_1 = 2$$

$$\omega = 0$$

$$Y_1 = \left(\frac{20}{(j\omega)^2 + 2j\omega + 10}\right)_{\omega=0} \cdot X_1$$

$$Y_1 = 2 \cdot 2 = 4$$

$$y_1(t) = 4$$

The second input:

$$x_2(t) = 3\cos(4t)$$

In phasor form

$$X_2 = 3 + j0$$

$$\omega = 4$$

$$Y_2 = \left(\frac{20}{(j\omega)^2 + 2j\omega + 10}\right)_{\omega=4} \cdot X_2$$

$$Y_2 = (-1.2 - j1.6) \cdot (3 + j0)$$

$$Y_2 = -3.6 - j4.8$$

$$y_2(t) = -3.6\cos(4t) + 4.8\sin(4t)$$

The third input

$$x_3(t) = 5\sin(6t)$$

Convert to phasors

$$X_3 = 0 - j5$$

$$\omega = 6$$

$$Y_3 = \left(\frac{20}{(j\omega)^2 + 2j\omega + 10}\right)_{\omega=6} \cdot X_3$$

$$Y_3 = (-0.6341 - j0.2927) \cdot (0 - j5)$$

$$Y_3 = -1.4634 + j3.1707$$

$$y_3(t) = -1.4634\cos(6t) - 3.1707\sin(6t)$$

The total output is then

$$y(t) = y_1(t) + y_2(t) + y_3(t)$$

$$y(t) = 4 - 3.6\cos(4t) + 4.8\sin(4t) - 1.4634\cos(6t) - 3.1707\sin(6t)$$

Note 1: A common mistake is to simplify the complex numbers

$$Y = Y_1 + Y_2 + Y_3 = (4) + (-3.6 - j4.8) + (-1.4634 + j3.1707)$$

$$Y = -1.0634 - 1.6293$$

This doesn't work:

- y1(t), y2(t), and y3(t) are all at different frequencies
- You can't simplify the sine waves

Note 2: If you prefer polar form, then

$Y_1 = 4$

$$y_1(t) = 4$$

$Y_2 = -3.6 - j4.8 = 6\angle = 126^0$

$$y_2(t) = 6\cos(4t + 126^0)$$

$Y_3 = -1.4634 + j3.1707 = 3.4921\angle 114^0$

$$y_3(t) = 3.9421\cos(6t + 114^0)$$

resulting in

$$y(t) = 4 + 6\cos(4t + 126^0) + 3.9421\cos(6t + 114^0)$$

Either answer is correct.