

ECE 341 - Homework #3

Dice Games and z-Transform.

Farkle

1) Compute the odds of rolling a 3 of a kind, 3 of a kind (two triplets) in Farkle

dice = xxx yyy

The number of ways you can roll 6 dice is

$$N = 6^6 = 46,656$$

The number of ways you can get two triplets is

$M = (6 \text{ numbers choose } 2) (6 \text{ spots for } x, \text{ choose } 3) (3 \text{ remaining spots for } y, \text{ choose } 3)$

$$M = \binom{6}{2} \binom{6}{3} \binom{3}{3} = 300$$

The odds of getting two triplets is

$$p = \left(\frac{M}{N} \right) = \left(\frac{300}{46,656} \right) = 0.00643$$

With a value of 2500 points, this adds to the expected value of rolling all six dice

$$2500 p = 16.075$$

Running a Monte-Carlo simulation in Matlab results in 647 / 100,000 cases of two triplets

$$p = 0.006470 \text{ (experimental)}$$

2) (option 1) Compute the odds of rolling 3 pair in Farkle.

dice = aa bb cc

M = (6 choose 3 for abc)(6 spots for a choose 2)(4 remaining spots for b choose 2)(2c2 for C)

$$M = \binom{6}{3} \binom{6}{2} \binom{4}{2} \binom{2}{2}$$

$$M = 1800$$

$$p = \left(\frac{1800}{46,656} \right) = 0.038580$$

2) (option 2) Compute the odds of rolling 3 of a kind in Farkle.

dice = (xxx aa b) or (xxx abc)

xxx aa b

M = (6c1 for x)(6c3 for spots for x)(5c1 for a)(3c2 for spots for a)(4c1 for c)(1c1 spot for c)

$$M_1 = \binom{6}{1} \binom{6}{3} \binom{5}{1} \binom{3}{2} \binom{4}{1} \binom{1}{1}$$

$$M_1 = 7200$$

xxx abc

M = (6c1 for x)(6c3 spots for x)(5c3 for abc)(3c1 spots for a)(2c1 spots for b)(1c1 spot for c)

$$M_2 = \binom{6}{1} \binom{6}{3} \binom{5}{3} \binom{3}{1} \binom{2}{1} \binom{1}{c}$$

$$M_2 = 7200$$

Add these together and you get 14,400 (same as enumeration)

$$p = \left(\frac{14,400}{46,656} \right) = 0.308642$$

z-Transforms

Assume X and Y have the following z-transforms

$$X = \left(\frac{1}{2}\right) \left(\frac{z+1}{z}\right) \quad \text{bernoulli trial (coin toss)}$$

$$Y = \left(\frac{1}{3}\right) \left(\frac{z^2+z+1}{z^2}\right) \quad \text{uniform distribution (3-sided die)}$$

3) Determine the z-transform and inverse z-transform for XX

XX is

$$XX = \left(\frac{1}{2}\right) \left(\frac{z+1}{z}\right) \cdot \left(\frac{1}{2}\right) \left(\frac{z+1}{z}\right)$$

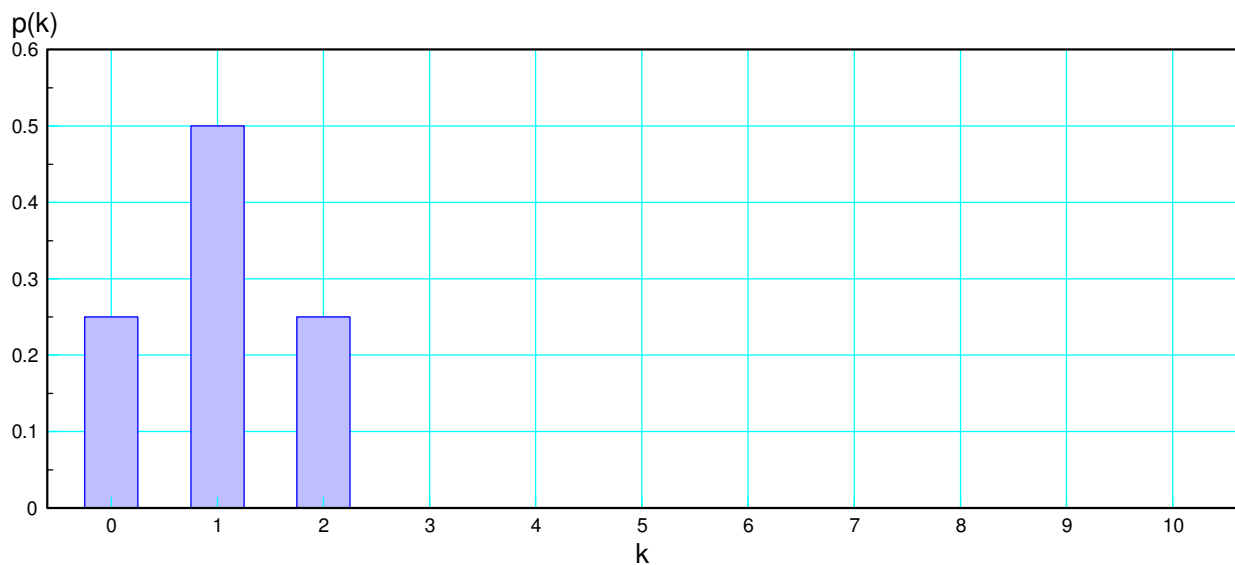
$$XX = \left(\frac{1}{4}\right) \left(\frac{z^2+2z+1}{z^2}\right) \quad \text{binomial distribution}$$

You can also write this as

$$XX = \left(\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)z^{-1} + \left(\frac{1}{4}\right)z^{-2}\right)$$

pdf: Apply the definition of z-transform

$$xx(k) = \left(\frac{1}{4}\right) \delta(k) + \left(\frac{1}{2}\right) \delta(k-1) + \left(\frac{1}{4}\right) \delta(k-2)$$



4) Determine the z-transform and inverse z-transform for XY

$$X = \left(\frac{1}{2}\right) \left(\frac{z+1}{z}\right) \quad \text{bernoulli trial (coin toss)}$$

$$Y = \left(\frac{1}{3}\right) \left(\frac{z^2+z+1}{z^2}\right) \quad \text{uniform distribution (3 sided die)}$$

z-transform of XY (also known as the moment generating function)

$$XY = \left(\frac{1}{6}\right) \left(\frac{z+1}{z}\right) \left(\frac{z^2+z+1}{z^2}\right)$$

$$XY = \left(\frac{1}{6}\right) \left(\frac{z^3+2z^2+2z+1}{z^3}\right)$$

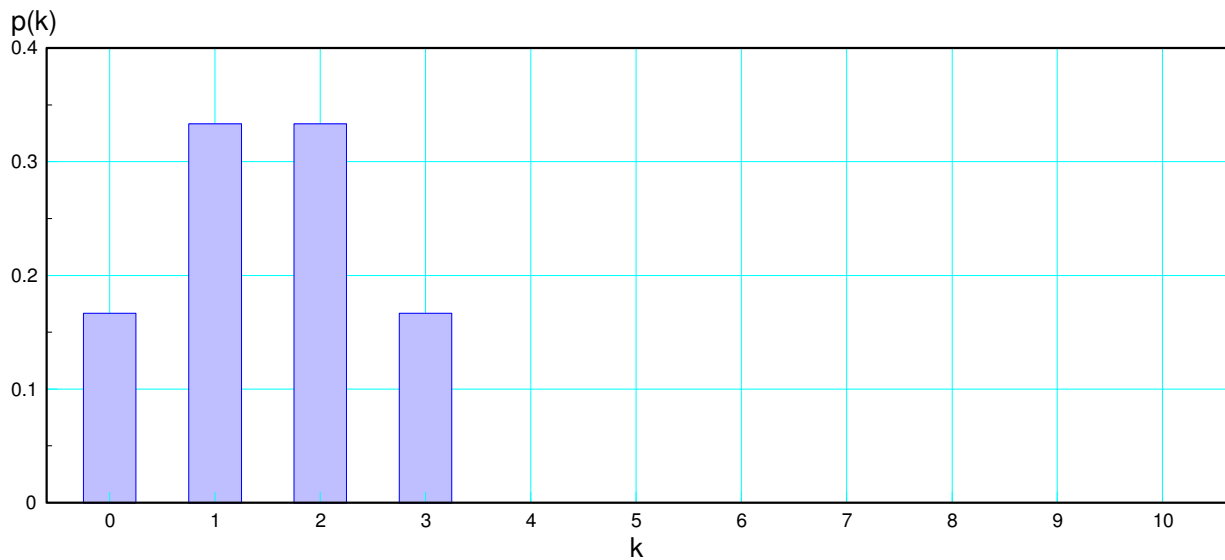
To take the inverse z-transform, simply apply the definition of z-transform

moment generating function:

$$XY = \left(\left(\frac{1}{6}\right) + \left(\frac{2}{6}\right)z^{-1} + \left(\frac{2}{6}\right)z^{-2} + \left(\frac{1}{6}\right)z^{-3}\right)$$

pdf

$$xy(k) = \left(\frac{1}{6}\right)\delta(k) + \left(\frac{2}{6}\right)\delta(k-1) + \left(\frac{2}{6}\right)\delta(k-2) + \left(\frac{1}{6}\right)\delta(k-3)$$



pdf for XY

5) Determine the z-transform and inverse z-transform of XY

$$X = 0.2 \left(\frac{z}{z-0.8} \right) \quad \text{geometric distribution}$$

$$Y = 0.5 \left(\frac{z}{z-0.5} \right) \quad \text{geometric distribution}$$

Solution: (moment generating function):

$$XY = 0.1 \left(\frac{z}{z-0.8} \right) \left(\frac{z}{z-0.5} \right) \quad \text{Pascal distribution}$$

Inverse z-transform (pdf)

$$XY = \left(\frac{0.1z}{(z-0.8)(z-0.5)} \right) z$$

$$XY = \left(\left(\frac{0.2667}{z-0.8} \right) + \left(\frac{-0.1667}{z-0.5} \right) \right) z$$

$$XY = \left(\frac{0.2667z}{z-0.8} \right) + \left(\frac{-0.1667z}{z-0.5} \right)$$

$$xy(k) = \left(0.26667(0.8)^k - 0.1667(0.5)^k \right) u(k)$$

