ECE 341 - Homework #11

Markov Chains.

Problem 1 & 2) Two teams, A and B, are playing a match made up of N games. For each game

- Team A has a 50% chance of winning
- There is a 20% chance of a tie, and
- Team B has a 30% chance of winning

In order to win the match, a team must be up by 2 games.

1) Determine the probabilty that team A wins the match after k games for k = {0 ... 10} using matrix multiplication.

First set up the state transistion matrix

$$z\begin{bmatrix} p2 \\ p1 \\ e \\ m1 \\ m2 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0.5 & 0 & 0 \\ 0 & 0.3 & 0.2 & 0.5 & 0 \\ 0 & 0 & 0.3 & 0.2 & 0 \\ 0 & 0 & 0 & 0.3 & 1 \end{bmatrix}\begin{bmatrix} p2 \\ p1 \\ e \\ m1 \\ m2 \end{bmatrix}$$

For A winning in 1..10 games

```
A = [1,0.5,0,0,0;0,0.2,0.5,0,0;0,0.3,0.2,0.5,0;0,0,0.3,0.2,0;0,0,0,0.3,1]
B = [0;0;1;0;0];
C = [1,0,0,0,0];
for k=1:10
    Awins = C * A^k * B;
    disp([k, Awins])
end
```

k	p(A)
1	0
2	0.2500
3	0.3500
4	0.4550
5	0.5230
6	0.5775
7	0.6170
8	0.6469
9	0.6692
10	0.6859

For A winning eventually, raise the state transistion matrix to a large power (say, 100 games)

```
A = [1,0.5,0,0,0;0,0.2,0.5,0,0;0,0.3,0.2,0.5,0;0,0,0.3,0.2,0;0,0,0,0.3,1]
```

A (one game)

1.0000	0.5000	0	0	0
0	0.2000	**0.5000**	0	0
0	0.3000	**0.2000**	0.5000	0
0	0	**0.3000**	0.2000	0
0	0	**0**	0.3000	1.0000

A^10 (10 games)

1.0000	0.8737	**0.6859**	0.4144	0
0	0.0137	**0.0249**	0.0228	0
0	0.0150	**0.0273**	0.0249	0
0	0.0082	**0.0150**	0.0137	0
0	0.0895	**0.2469**	0.5242	1.0000

A^100 (100 games)

1.0000	0.9007	**0.7353**	0.4596	0
0	0.0000	**0.0000**	0.0000	0
0	0.0000	**0.0000**	0.0000	0
0	0.0000	**0.0000**	0.0000	0
0	0.0993	**0.2647**	0.5404	1.0000

Assuming you start out at even, A has a 73.53% chance of winning the match.

2) Determine the z-transform for the probability that A wins the match after k games
   - From the z transforms, determine the explicit function for p(A) wins after game k.

$$zX = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0.5 & 0 & 0 \\ 0 & 0.3 & 0.2 & 0.5 & 0 \\ 0 & 0 & 0.3 & 0.2 & 0 \\ 0 & 0 & 0 & 0.3 & 1 \end{bmatrix} X \qquad X(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Y = A_{wins} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} X$$

```
X0 = [0;0;1;0;0]

     0
     0
     1
     0
     0

C = [1,0,0,0,0]

    1      0      0      0      0

G = ss(A, X0, C, 0, 1);
zpk(G)

           0.25 (z-0.2)
  ------------------------------------
  (z-1) (z-0.7477) (z-0.2) (z+0.3477)

Sampling time (seconds): 1
```

Multiply by z to get the z-transform for A winnings

$$Y(z) = \left(\frac{0.25}{(z-1)(z-0.7477)(z+0.3477)}\right) z$$

A property of z-transforms is

The delay is the difference in order from the denominator to the numerator

Multiply by z^2 (not necessary but simplifies the answer)

$$z^2 Y = \left(\frac{0.25z^3}{(z-1)(z-0.7477)(z+0.3477)}\right)$$

Factor out a z

$$z^2 Y = \left(\frac{0.25z^2}{(z-1)(z-0.7477)(z+0.3477)}\right) z$$

Do partial fractions

$$z^2 Y = \left(\left(\frac{0.7352}{z-1} \right) + \left(\frac{-0.5057}{z-0.7477} \right) + \left(\frac{0.0205}{z+0.3477} \right) \right) z$$

$$z^2 Y = \left(\frac{0.7352z}{z-1} \right) + \left(\frac{-0.5057z}{z-0.7477} \right) + \left(\frac{0.0205z}{z+0.3477} \right)$$

Take the inverse z-transform

$$z^2 y(k) = \left(0.7352 - 0.5057(0.7477)^k + 0.0205(-0.3477)^k \right) u(k)$$

Divide by z^2 (meaning delay 2 samples)

$$\boxed{y(k) = \left(0.7352 - 0.5057(0.7477)^{k-2} + 0.0205(-0.3477)^{k-2} \right) u(k-2)}$$

Comparing the two answers (problem #1 and #2)

```
A = [1,0.5,0,0,0;0,0.2,0.5,0,0;0,0.3,0.2,0.5,0;0,0,0.3,0.2,0;0,0,0,0.3,1]
B = [0;0;1;0;0];
C = [1,0,0,0,0];
for k=1:10
    Awins = C * A^k * B;
    Az = (0.7352 - 0.5057*(0.7477)^(k-2) + 0.0205*(-0.3477)^(k-2) ) * (k>=2);

    disp([k, Awins, Az])
end
```

k	Prob1	Prob2
1	0	0
2	0.2500	0.2500
3	0.3500	0.3500
4	0.4550	0.4550
5	0.5230	0.5230
6	0.5775	0.5774
7	0.6170	0.6169
8	0.6469	0.6469
9	0.6692	0.6691
10	0.6859	0.6858

3) Two players are playing a game of tennis. To win a game, a player must win 4 points *and* be up by 2 points.

- If player A reaches 4 points and player B has less than 3 points, the game is over and player A wins.
- If player A reaches 4 points and player B has 3 points, then the game reverts to 'win by 2' rules. Both players keep playing until one of them is up by 2 games.

Supppose:

- Player A has a 60% chance of winning any given point
- Player B has a 40% chance of winning any given point.

What is the probabilty that player A wins the game (first to 4 games, win by 2)?

- Note: This is a combination of a binomial distribution (A has 4 points while B has 0, 1, or 2 points) along with a Markov chain (A and B both have 3 points, at which point it becomes a win-by-2 series)

The ways A can win are (A always has to win the last game)

a) 4 - 0 A wins after being up 3 - 0

b) 4 - 1 A wins after being up 3 - 1

c) 4 - 2 A wins after being up 3 - 2

d) 3 - 3 A wins the Markov chain (win-by-two series)

This is likewise a conditional probability

$$p(A) = p(A|a)p(a) + p(A|b)p(b) + p(A|c)p(c) + p(A|d)p(d)$$

The odds are

a) 4 - 0: This is a binomial problem

$$p(A|a) = 1$$ *the match is over is A is up 4-0*

$$p(a) = \left(\binom{3}{3} p^3 q^0 \right) \cdot p = 0.1296$$

b) 4 - 1: Again a binomial (3 - 1 followed by A winning)

$$p(A|b) = 1$$ *the match is over if A is up 4-1*

$$p(b) = \left(\binom{4}{3} p^3 q^1 \right) \cdot p = 0.2047$$

c) 4 - 2: A us up (3 - 2) then wins

$$p(A|c) = 1$$ *the match is over if A is up 4-2*

$$p(c) = \left(\binom{5}{3} p^3 q^2 \right) \cdot p = 0.2047$$

d) 3 - 3: Ending up at duce:

$$p(d) = \left(\binom{6}{3} p^3 q^3 \right) = 0.2765$$

The chance of A winning after being at duce is the solution to a Markov chain:

$$y(k+1) = \begin{bmatrix} 1 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix} y(k)$$

```
A = [1,0,0,0,0 ; 0.6,0,0.4,0,0 ; 0,0.6,0,0.4,0 ; 0,0,0.6,0,0.4 ; 0,0,0,0,1]'

    1.0000    0.6000         0         0         0
         0         0    0.6000         0         0
         0    0.4000         0    0.6000         0
         0         0    0.4000         0         0
         0         0         0    0.4000    1.0000

A^100

    1.0000    0.8769    0.6923    0.4154         0
         0    0.0000         0    0.0000         0
         0         0    0.0000         0         0
         0    0.0000         0    0.0000         0
         0    0.1231    0.3077    0.5846    1.0000
```

A has a 69.23% chance of winning if you start at duce.

$$p(A|d) = 0.6823$$

Put it all together

$$p(A) = p(A|a)p(A) + p(A|b)p(b) + p(A|c)p(c) + p(A|d)p(d)$$

$$p(A) = (1)(0.1296) + (1)(0.2047) + (1)(0.2047) + (0.6923)(0.2765)$$

$$p(A) = 0.7357$$

Player A has a 73.57% chance of winning the game