# Combinations and Permutations 5-Card Stud Poker

# ECE 341: Random Processes Lecture #4

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

#### **Combinations and Permutations: 5-Card Stud**

• Combinatorics: how many ways an event can happen.

Definitions:

- n! "n factorial" n x (n-1) x (n-2) x ... x 2 x 1
- 0! = 1 Just define zero factorial to be one.
- p(x) "the probability of outcome x"
- ${}_{n}P_{m}$  "Permutations of n events taken m at a time".  ${}_{n}P_{m} = \frac{n!}{(n-m)!}$
- ${}_{n}C_{m}$  "Combinations of n events taken m at a time"  ${}_{n}C_{m} = \frac{n!}{m! \cdot (n-m)!}$ •  $\binom{n}{m}$  "n choose m". Another way of writing  ${}_{n}C_{m}$
- Sample With Replacement: Each sample is of the same population size
- Sample Without Replacement: Each time you sample, the remaining population becomes one smaller

#### Monte-Carlo vs. Enumeration vs. Combinatorics

#### Monte-Carlo works

- Play a game N times & count the number of successes
- Results are approximate
- Matlab is a great tool for Monte-Carlo simulations

#### Enumeration works

- List out all possibilities and count the number of successes
- Answers are exact
- Matlab is a great tool for enumeration
  - Lots of nested for-loops

#### Combinatorics

- Calculate the number of ways to get a given outcome
- Answers are also exact (but can be tricky to find)

## Permutations

- Sample without replacement
- Order matters

#### Example: Volleyball Team

- How many teams can you make with 20 players?
- Each player is assigned to a position (order matters)

#### Solution

- 20 ways to pick the fist player
- 19 ways to pick the second...

 $N = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15$ 

N = 27,907,200

You can also write this as "20 Pick 6"

$$N = \left(\frac{20!}{(20-6)!}\right) =_{20} P_6$$



# Combinations

- Sample without replacement
- Order doesn't matter

#### Example: Volleyball Team

- How many different volleyball teams can you make with 20 players?
- Teams can shuffle positions (order doesn't matter)

#### Solution

• Each permutation overcounts each time by 6!

$$N = \frac{{}_{20}P_6}{6!} = 38,760$$

You can also write this as 20 Choose 6:

$$N =_{20} C_6 = \begin{pmatrix} 20\\6 \end{pmatrix} = \left(\frac{20!}{(20-6)!(6)!}\right) = 38,760$$



#### **Poker: 5-Card Stud**

- Starting out, a deck of 52 cards is shuffled.
- The first card is played face down so that only the player sees this card.
- The second card is played face up so all players can see it. After two cards are played, bets are made.
- Once betting stops, a third card is played face up and betting starts over again.
- Ditto for the 4th card and 5th card.
- Once betting is finished with 5 cards for each player, the face down card is revealed and the winner is determined.



#### **Rank of Hands**

The winning hands (in order) for poker are

- Royal Flush: 10-J-Q-K-A of the same suit
- Straight-Flush: A run of 5 cards in the same suit
- 4 of a kind: Four of your cards have the same value. Ex: J-J-J-J-x
- Full-House: 3 of a kind and a pair. Ex: J-J-J-Q-Q
- Flush: All cards of the same suit
- Straight: A run of 5 cards
- 3 of a kind: Three of your cards match. Ex: J-J-J-x,y
- 2-Pair: Two pairs of cards. J-J-Q-Q-x
- Pair: Two cards match. J-J-x-y-z
- High-Card: Other. No pairs, no straight, no flush.

# Probability of Each Type of Hand (take 1)

• Wikipedia

POKER HAND		COUNT	Odds Against	
2	Straight Flush	40	64,974	
3	Four of a Kind	624	4,165	
4	Full House	3,744	694.17	
5	Flush	5,108	508.8	
6	Straight	10,200	254.8	
7	Three of a Kind	54,912	47.33	
8	Two Pair	123,552	21.04	
9	One Pair	1,098,240	2.37	
10	High Card	1,302,540	2	
Total		2,598,960		

# Probability of Each Type of Hand (take 2)

- Enumeration
- Nested for-loops in Matlab
- Takes about 30 minutes to go through all possible poker hands (2.5 million)

POKER HAND		COUNT	Odds Against	
2	Straight Flush	40	64,974	
3	3 Four of a Kind 624		4,165	
4	4 Full House 3,744		694.17	
5	Flush	5,108	508.8	
6	Straight	10,200	254.8	
7	Three of a Kind	54,912	47.33	
8	Two Pair	123,552	21.04	
9	One Pair	1,098,240	2.37	
10	High Card	1,302,540	2	
Total		2,598,960		

# Probability of Each Type of Hand (take 3)

• Combinatorics

Compute the number of ways to get each type of hand

- Answers should match enumeration (exact answers)
- Takes lots less time
- Kind of tricky

# **Factorials:**

How many ways are there to shuffle a deck of 52 cards?

- Select the first card from 52 possibilities
- The second card from 51 possibilities
- The third card from 50 possibilities.

• etc.

The total number of ways a deck can be shuffled is

 $N = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \dots 2 \cdot 1$ 

This is 52 factorial, written as

*N* = 52!

Most calculators have a factorial key

 $N = 8.0658 \cdot 10^{67}$ 

Note that this is too many combinations for even a computer to run through.

# **Permutations: (Order Matters)**

The number of ways a given hand can play out is..

- Select the first card from 52 possibilities
- The second card from 51 possibilities
- The third card from 50 possibilities.

giving

 $N = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ 

Another way to write this is

$$N =_{52} P_5 = \left(\frac{52!}{(52-5)!}\right) = 311,875,200$$

There are 311,875,20 different ways a given hand can play out.

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# Combinations

• order doesn't matter

When determining who won the hand, the order of the cards doesn't matter.

- Select the first card from 5 possibilities
- The second card from 4 possibilities
- The third card from 3 possibilities.

#### or

$$M = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$M = 5! = 120$$

The number of hands in poker is then

$$_{52}C_5 = \begin{pmatrix} 52\\5 \end{pmatrix} = #hands = \begin{pmatrix} 52!\\(52-5)!\cdot 5! \end{pmatrix} = 2,598,960$$



# **Probability of a Royal Flush**

• order doesn't matter

### The probability of any given hand as

 $p(x) = \frac{\text{the total number of hands that are x}}{2,598,960}$ 

If you can compute the number of ways of getting a hand, you know the probability

From enumeration, N = 4

- Four ways to get a royal flush
- How to compute?



# **Probability of a Royal Flush**

Use enumeration

- There are only four possibilities
  - Spades, hearts, clubs, and diamonds
- $p(\text{royal flush}) = \left(\frac{4}{2,598,960}\right) = \frac{1}{649,740}$

The odds against getting a royal flush are 649,740 : 1

You should on average get a royal flush once every 649,740 hands.



# 4 of a kind:

From enumeration, there are 624 ways to get a 4-of-a-kind

• How to compute?



# 4 of a kind (cont'd)

There are several ways to compute this:

Assume the cards in your hand are x-x-x-y

- There are 13 cards in a deck: choose one
- Of the four cards of that value, choose four
- Of the 48 remaining cards, choose one

$$N = \begin{pmatrix} 13 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 48 \\ 1 \end{pmatrix} = 624$$

There are 624 different hands that give you a 4-of-a-kind.

• The odds being dealt 4 of a kind is 4165 : 1

$$p(N) = \left(\frac{624}{2,598,960}\right) = \left(\frac{1}{4165}\right)$$



#### 3-of-a-kind

• Enumeration: N = 54,912



## 3-of-a-kind: xxx a b (N = 54,912)

- Of the 13 value, choose one for x
- Of the four cards of that type, choose three
- Of the 12 remaining values, choose two for a & b
  - When elements have the same frequency, do together
- Of the four a's, choose 1
- Of the four b's, choose 1

$$N = \begin{pmatrix} 13 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 12 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 54,912$$

The odds are 47.33 : 1 against

 $p(N) = \left(\frac{54,912}{2,598,960}\right)$ 



#### Flush

- Five cards of one suit
- From Wikipedia, N = 5108



# Flush (N = 5108)

- Of the four suits, choose one
- Of the 13 cards in that suit, choose five  $N = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 13 \\ 5 \end{pmatrix} = 5,148$
- This includes straight-flushes (N = 40).
  - Removing these gives 5,108 different flushes

#### The odds 504.8 : 1 against

$$p(N) = \left(\frac{5,108}{2,598,960}\right) = \left(\frac{1}{504.8}\right)$$



# Straight

- A run of five cards
- 2 or more different suits
- Enumeration: N = 10,200



# Straight (N = 10,200)

- A straight can start with an Ace through a 10.
  - That gives 40 starting cards (10 values in 4 suits)
- Of the four cards of the next value, choose one
- Of the four cards of the next value, choose one

SO

$$N = 40 \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 10,240$$



This also counts straight-flushes (40), so the total number of straights is N = 10,200

giving the probability of 254.8:1 against

### 2-Pair

- Hand = xx yy z
- N = 123,552 (enumeration)



# 2-Pair (N = 123,552)

- Of the 13 values (Ace through King), pick two
- For the first value, pick two cards of the four in the deck
- For the second value, pick two cards of the four in the deck
- For the last card, pick one (44 choose 1)

 $N = \begin{pmatrix} 13 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 44 \\ 1 \end{pmatrix}$ 

N = 123,552

The odds of getting 2-pair are

 $p(x) = \left(\frac{123,552}{2,598,960}\right) = \left(\frac{1}{21.035}\right)$ 

or 21.03 : 1 odds against getting two-pair.



#### Pair (N = 1,098,240)

• xx a b c

N = (13 choose 1 for x) \* (4 x's choose 2)

\* (12 choose 3 for a/b/c) \* (4a's choose 1) \* (4b's choose 1)\*(4c's choose 1))

$$N = \begin{pmatrix} 13\\1 \end{pmatrix} \begin{pmatrix} 4\\2 \end{pmatrix} \cdot \begin{pmatrix} 12\\3 \end{pmatrix} \begin{pmatrix} 4\\1 \end{pmatrix} \begin{pmatrix} 4\\1 \end{pmatrix} \begin{pmatrix} 4\\1 \end{pmatrix} \begin{pmatrix} 4\\1 \end{pmatrix}$$
$$N = 1,098,240$$

The odds of being dealt a pair is 2.366 : 1 against

$$p(x) = \left(\frac{1,098,240}{2,598,960}\right) = \left(\frac{1}{2.366}\right)$$

#### Probability of all poker hands

https://allmathconsidered.wordpress.com/2017/05/23/the-probabilities-of-poker-hands/

POKER HAND		COUNT	Odds Against	
2	Straight Flush	40	64,974	
3	Four of a Kind	624	4,165	
4	Full House	3,744	694.17	
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Total		2,598,960		

#### Challenge: What are the odds if you play with two decks

- Two 52-card decks
- 104 cards total

How many different poker hands are there?

- Deal 5 cards
- Order doesn't matter



What are the odds of 4-of-a-kind?

• 104-card deck



4-of-a-Kind with Two Decks

# Hands

$$M = \left(\begin{array}{c} 104\\5 \end{array}\right) = 91,962,520$$

# 4-of-a-kind (xxxx y)  

$$N = \begin{pmatrix} 13 \\ 1 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 1 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

$$N = 87,360$$

In 100,000 hands, you should get 94.99 4-of-a-kind hands

$$n = \left(\frac{87,360}{91,962,520}\right) 100,000 = 94.99$$

What are the odds of a full-house?

• 104-card deck



What are the odds of a full-house?

# Hands

$$M = \left(\begin{array}{c} 104\\5 \end{array}\right) = 91,962,520$$

# Full-House (xxx yy)  $N = \begin{pmatrix} 13 \\ 1 \end{pmatrix} \begin{pmatrix} 8 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 1 \end{pmatrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix}$  N = 244,608

In 100,000 hands, you expect 266 full-houses

$$n = \left(\frac{244,608}{91,962,520}\right) \cdot 100,000$$
$$n = 265.98$$

# How to check your answer?

#### Enumeration takes too long

- 30 minutes for a 52-card deck
- 16 hours for a 104-card deck

Monte-Carlo will get you close

```
5 -
        Pair4 = 0;
 6 -
        FullHouse = 0;
        Pair3 = 0;
 7 -
        Pair22 = 0;
 8 -
 9 -
        Pair2 = 0;
10
11 -
      - for i0 = 1:1000
12
13 -
        X = rand(1, 52);
14 -
        [a,Deck] = sort(X);
15 -
        Hand = Deck(1:5);
       Value = mod(Hand, 13) + 1;
16 -
        Suit = floor(Hand/13) + 1;
17 -
18
19 -
       N = zeros(1, 13);
      \bigcirc for n=1:13
20 -
21 -
            N(n) = sum(Value == n);
22 -
        end
23
24 -
        [N,a] = sort(N);
25
26 -
        if (N(13) == 4) Pair4 = Pair4 + 1; end
27 -
       if ((N(13) == 3)*(N(12) == 2)) FullHouse=FullHouse +
       if ((N(13) == 3)*(N(12) < 2)) Pair3 = Pair3 + 1; end
28 -
       if ((N(13) == 2)*(N(12) == 2)) Pair22 = Pair22 + 1;
29 -
       if ((N(13) == 2)*(N(12) < 2)) Pair2 = Pair2 + 1; end
30 -
31
32 -
        end
```

#### **Monte-Carlo Simulations**

Monte-Carlo code to play poker with two decks

- A 104-card deck is shuffled
- A 5-card hand is dealt
- The type of hand it is then logged
- This experiment is then repeated a large number of times

The odds are then approximately the number of times each hand occurred divided by the number of hands dealt.

#### **Monte-Carlo Procedure:**

#### Shuffle two decks of cards (52 cards in each deck):

X = rand [a,Deck] Deck = De	(1,52*2) = sort eck - 1;	); (X);							
18	69	70	47	65	41	14	85	7	22
Take each o Deck = mo Deck(1:10 18	card mo od (Deck, D) 17	od 52 , 52); 18	47	13	41	14	33	7	22
Draw a 5-c >> Hand =	ard han = Deck(2	<b>d</b> 1:5)							

Hand = 18 17 18 47 13

#### Determine the value and suit of each card:

>> Value = mod(Hand, 13) + 1

Value = 6 5 6 9 1 >> Suit = floor(Hand/13) + 1 Suit = 2 2 2 4 2

#### Your hand is

- 6 of diamonds
- 5 of diamonds
- 6 of diamonds
- 9 of spades
- Ace of diamonds

For pairs, count the frequency of each card

```
N = zeros(1,13);
for n=1:13
     N(n) = sum(Value == n);
end
```

Then determine what kind of hand by

- Sorting the number of matching cards
- Checking if the maximum of the matches is 4 (4-of-a-kind), 3 and 2 (full house), etc

```
[N,a] = sort(N, 'descend');
if (N(1) == 4) Pair4 = Pair4 + 1; end
if ((N(1) == 3)*(N(2) == 2)) FullHouse=FullHouse + 1; end
if ((N(1) == 3)*(N(2) < 2)) Pair3 = Pair3 + 1; end
if ((N(1) == 2)*(N(2) == 2)) Pair22 = Pair22 + 1; end
if ((N(1) == 2)*(N(2) < 2)) Pair2 = Pair2 + 1; end</pre>
```

#### Matlab Code

```
Pair4 = 0;
Initialize counters
                                             FullHouse = 0;
                                             Pair3 = 0;
                                             Pair22 = 0;
                                             Pair2 = 0;
repeat 100,000 times
                                             tic
                                             for i0 = 1:1e5
shuffle two decks
                                                 X = rand(1, 52*2);
                                                 [a, Deck] = sort(X);
                                                 Deck = Deck - 1;
deal out 5 cards
                                                 Deck = mod(Deck, 52);
                                                 Hand = Deck(1:5);
                                                 Value = mod(Hand, 13) + 1;
determine values & suits
                                                 Suit = floor (Hand/13) + 1;
                                                 N = zeros(1, 13);
count frequency of each card
                                                 for n=1:13
                                                    N(n) = sum(Value == n);
                                                 end
check of each type of hand
                                                 [N,a] = sort(N, 'descend');
                                                 if (N(1) == 4) Pair4 = Pair4 + 1; end
                                                 if ((N(1) == 3)*(N(2) == 2)) FullHouse=FullHouse + 1;
                                                 if ((N(1) == 3)*(N(2) < 2)) Pair3 = Pair3 + 1; end
                                                 if ((N(1) == 2)*(N(2) == 2)) Pair22 = Pair22 + 1; end
                                                if ((N(1) == 2)*(N(2) < 2)) Pair2 = Pair2 + 1; end
                                             end
                                             toc
                                             disp('4 of a kind, Full House, 3 of a kind, 2 Pair, Pair'
                                             disp([Pair4, FullHouse,Pair3,Pair22,Pair2])
```

#### **Monte-Carlo Results**

- One Deck (known odds)
- Always test your code

#### The results for 1 million hands of poker:

	4-of-kind	full-house	3-o-k	2-pair	pair
X =	252	1409	21097	47498	423055

#### The odds match up (per 1,000,000 hands)

Hand	Computed	Simulated
4 of a kind	240.1	252
Full-House	1,440.9	1,409
3 of a kind	21,141	21,097
2-pair	47,619	47,489
pair	421,940	423,055

#### **Monte-Carlo Results**

- Monte-Carlo results match up with calculations
- Playing with two decks affects the odds

	One Deck 100,000 hands	Two Decks		
Hand	Calculations	Calculations	Simulated	
4 of a kind	24	94.99	101	
Full-House	144.1	265.98	295	
3 of a kind	2,114.1	?	3,357	
2-pair	4,761.9	?	5,752	
pair	42,194	?	44,485	

# Summary

With combinatorics, you can calculate how many combinations result in a given outcome

- 4 of a kind
- full house
- 3 of a kind

Where there are a large number of possibilities, this cans save a lot of time

• Enumeration can take hours or days to go through every permutation