Dice Games: Farkle

ECE 341: Random Processes

Lecture #6

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Dice Games: Farkle

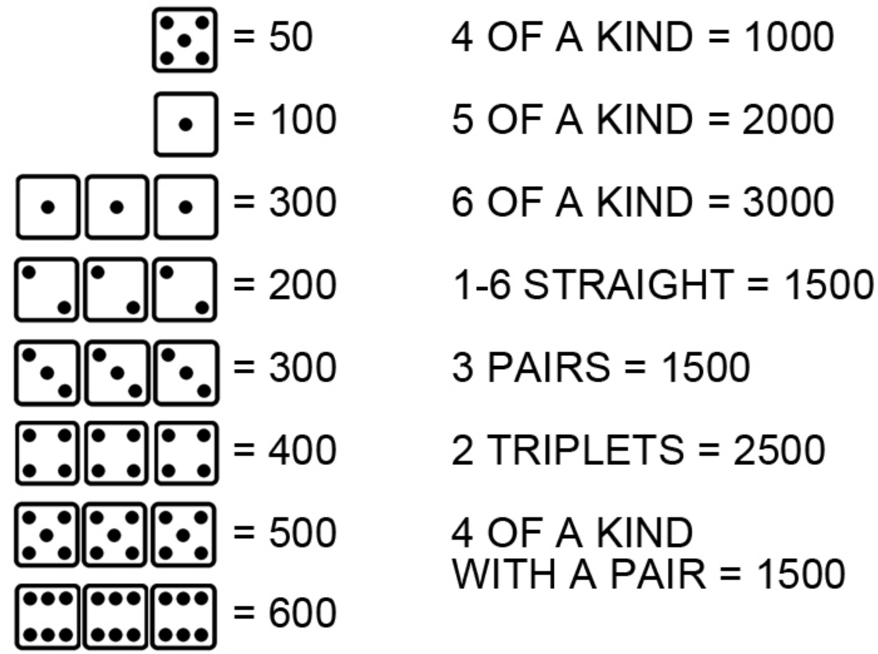
- Enumeration and tree analysis also apply to dice games such as Farkle

The rules of Farkle are simple:

- Start the game by tossing six dice. You can then select any of the dice that score points to keep. If you score zero points on a toss, you say "Farkle," your turn is over, and you score zero points for that round.
- If you do score points, you may take the remaining dice and toss them. Again, if you do, you must score points on the dice you tossed (and keep) - otherwise your turn is over and you score zero points.
- If all six of your dice score points and you have zero dice left, you start over with six dice and keep going.
- Once one player reaches 10,000 points, each other player gets one more turn. After that, highest score wins.

The ways to score points are:

- Each one is worth 100 points
- Each five is worth 50 points
- Three of a kind: 100 times the die value
- 4 of a kind: 1000 points
- 5 of a kind: 2000 points
- 6 of a kind: 3000 points
- 1-6 Straight: 1500 points
- Three pair: 1500 points
- Four of a kind and a pair: 1500 points
- Two triplets: 2500 points



Farkle Strategy

- When do you keep rolling?
- When do you quit and take your points?

If you stop rolling, you keep the points you scored that round.

- but you give up any points you might have scores if you kept rolling

If you elect to toss the dice, you risk scoring zero points (a Farkle) and losing everything you scored that round.

With that, you can compute several odds.

Odds of Scoring Points when Tossing One Die

This is pretty easy: there are six possible die rolls:

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{ 1, 2, 3, 4, 5, 6 }
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Two of the six score points. The odds are then

- 2/6 Chance of scoring points
- 4/6 Chance of a Farkle

Should you toss one die?

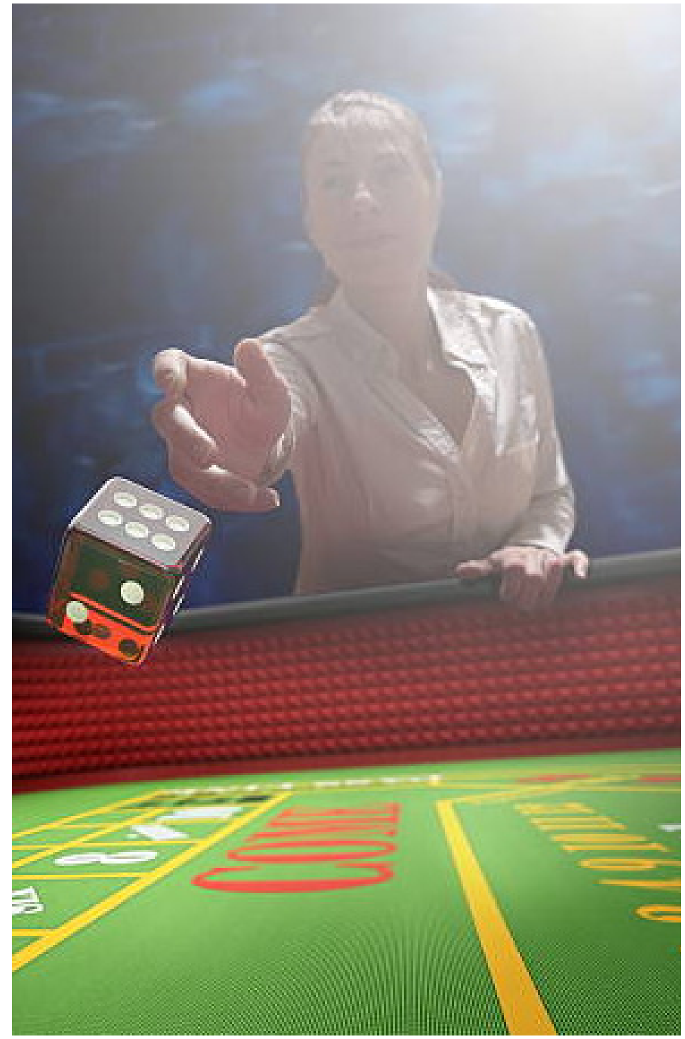
The expected return is equal to

- The points you expect to get if you are successful
- Times the probability of success

minus

- The points you expect to lose if you are not successful
- Times the probability of failure

Assume for now that the expected score when tossing six dice is 300 points



The expected return is

- E(return) = (1/6) * (100 points + 300 points) roll a 1 and then roll 6 dice
- + (1/6) * (50 points + 300 points) roll a 5 and then roll 6 dice
- - (4/6) * X lose all X points if you Farkle
- E(return) = 125 - 4/6 X

The expected return is positive as long as X is less than 187 points

- If you have less than 187 points, it's worth while to toss that last die.
- Otherwise, take your points and end your turn.)

Odds of scoring points: two dice:

In this case, there are 36 possibilities.

- The rolls that score points are shown in red:
- All dice score are shown in blue

11	**12**	**13**	**14**	**15**	**16**
21	22	23	24	**25**	26
31	32	33	34	**35**	36
41	42	43	44	**45**	46
51	**52**	**53**	**54**	**55**	**56**
61	62	63	64	**65**	66

There are 20 ways to score points when tossing two dice

- 4/36 both dice score (and you get to toss 6 dice next roll: shown in blue)
- 16/36 one die scores (shown in red)
- 16/36 Farkle

The expected return for tossing two dice is then

- E(return) = (1/36) * (200 points + 300 points)   roll two 1's
- + (2/36) * (150 points + 300 points)   roll 1 & 5
- + (1/36) * (100 points + 300 points)   roll two 5's
- + (8/36) * (100 points)   roll a single 1
- + (8/36) * (50 points) roll a single 5
- - (16/36) * X Farkle

or

- E(return) = 83.333 - 16/36 * X

The expected return is positive if $X < 187$, meaning

- You should toss two dice if your score this round is less than 187 points.
- You should end your turn if you score this round is more than 187 points

Odds when tossing three dice

There are 216 possible outcomes when tossing three dice. The odds of not rolling any 1's or 5's are:

$$p = \left(\frac{4}{6}\right)\left(\frac{4}{6}\right)\left(\frac{4}{6}\right) = 0.2963$$

The number of ways you can toss 3 dice and get no 1's or 5's is thus

$$M = 0.2963 \cdot 216 = 64$$

There are actually four more ways to score (three 2's, three 3's, etc) for

$$M = 64 - 4 = 60$$

meaning the number of ways to score when rolling three dice is

$$N = 216 - M = 156$$

This gives

- 156 / 216 the probability of scoring when tossing three dice
- 60 / 216 the probability of a Farkle

Expected Return when Tossing Three Dice:

Three of a kind: 100 * die value + 300 points (all dice score)

- E(return) = (1/216) * (300 + 300) three 1's
- + (1/216) * (200 + 300) three 2's
- + (1/216) * (300 + 300) three 3's
- + (1/216) * (400 + 300) three 4's
- + (1/216) * (500 + 300) three 5's
- + (1/216) * (600 + 300) three 6's

Add to this the chance of rolling

- + (3/216) * (250 + 300) 115 (3 permutations)
- + (3/216) * (200 + 300) 155 (3 permutations)
- + (12/216) * (200) 11x (12 permutations)
- + (12/216) * (150) 15x (12 permutations)
- + (12/216) * (100) 55x (12 permutations)
- + (48/216) * (100) 1xx
- + (48/216) * (50) 5xx
- - (60/216) * X Farkle

The expected return is thus

E(return) = 91.891 - (60/216)*X

The expected return is positive if X is less than 330

- You should roll three dice if you have less than 330 points in this round
- You should stop rolling the dice if you have more than 330 points

Farkle & Enumeration

- assume you're rolling six dice...

Type of Roll	# Occurances
6 of a kind	6
5 of a kind	180
4 of a kind + 2 of a kind	450
4 of a kind	1,800
two tripples	300
3 of a kind	14,400
three pairs	1,800

- Can we get the same result using combinatorics?

C:\Documents and Settings\Administrator\My Documents\MATLAB\ECE341\04 Farkle\Farkle2.m

```
for d1=1:6
  for d2=1:6
    for d3=1:6
      for d4=1:6
        for d5=1:6
          for d6 = 1:6
            Dice = [d1,d2,d3,d4,d5,d6];
            Dice = sort(Dice);

            N = zeros(1,6);
            for i=1:6
                N(i) = sum(Dice == i);
            end
            [N,b] = sort(N, 'descend');

            if (N(1) == 6) Pair6 = Pair6 + 1; end
            if (N(1) == 5) Pair5 = Pair5 + 1; end
            if ((N(1)==4)*(N(2)==2)) Pair42 = Pair42 + 1; end
            if ((N(1)==4)*(N(2)<2))  Pair4 = Pair4 + 1; end
            if ((N(1)==3)*(N(2)==3)) Pair33 = Pair33 + 1; end
            if ((N(1)==3)*(N(2)<3)) Pair3 = Pair3 + 1; end
            if ((N(1)==2)*(N(2)==2)*(N(3)==2)) Pair222 = Pair222+1; end
            end
          end
        end
      end
    end
end

disp('      xxxxxx      xxxxxy    xxxxyz    xxxxyy     xxxabc    xxy
```

6-of-a-Kind

- assume rolling all six dice
- 6-possible ways using enumeration
- Come up with the same answer (6) using combinatorics

Solution: xxxxxx

- (6 values choose 1 for x)(6 spots, choose all 6)

$$M = \binom{6}{1}\binom{6}{1} = 6$$

5-of-a-kind

- xxxxx y
- assume rolling all six dice
- 180 possible ways using enumeration
- Calculate this using combinatorics

5-of-a-kind (solution)

- xxxxx y
- M = 180 from enumeration
- (6 values choose 1 for x)(6 spots for x, choose 5)(5 other values, choose 1 for y)

$$M = \binom{6}{1}\binom{6}{5}\binom{5}{1} = 180$$

4 of a kind

- xxxx y z
- assume rolling all six dice
- M = 1800 from enumeration
- Calculate this using combinatorics

4-of-a-Kind (Solution)

- xxxx y z
- M = 1800 from enumeration

M = (6 values choose 1 for x)(6 spots choose 4 for x)

* (5 remaining numbers pick 1 for y)

$$M = \binom{6}{1}\binom{6}{4}\binom{5}{1}\binom{4}{1}$$

$$M = 1800$$

3-of-a-Kind

What is the probability of rolling 3-of-a-kind with 6 dice?

- xxx aa b
- xxx abc

From enumeration, M = 14,400

3-of-a-kind (cont'd)

xxx a b c

M = (6 values choose 1 for x)(6 spots choose 3 for x) *

(5 remaining values choose 1 for a) *

(4 remaining values choose 1 for b) *

(3 remaining values choose 1 for c)

$$M = \binom{6}{1}\binom{6}{3}\binom{5}{1}\binom{4}{1}\binom{3}{1}$$

$$M = 7200$$

3-of-a-kind (cont'd)

Dice = xxx aa b

M = (6 values choose 1 for x)(6 spots choose 3 for x)

* (5 remaining values choose 1 for a)(3 spots choose 2 for a)

* (4 remaining values choose 1 for b)(1 spot for b choose 1)

$$M = \binom{6}{1}\binom{6}{3}\binom{5}{1}\binom{3}{2}\binom{4}{1}\binom{1}{1}$$

$$M = 7,200$$

Add the two together and you get 14,400

- same as enumeration

Two Tripples

What are the odds of rolling two triples?

- xxx yyy
- Enumeration: M = 300

Two Triples

Since x and y have the same frequency, analyze these together

- M = (6 values choose 2 for x and y)
- * (6 spots for x, choose 3)
- * (3 remaining spots for y choose 3)

$$M = \binom{6}{2}\binom{6}{3}\binom{3}{3} = 300$$

Three Pairs

What are the odds of rolling three pairs?

- xx yy zz
- Enumeration: M = 1800

Three Pairs

Since x, y, and z have the same frequency, analyze these together

- M = (6 values choose 3 for xyz)
- * (6 spots for x choose 2)
- * (4 remaining spots for y choose 2)
- * (2 remaining spots for z choose 2)

$$M = \binom{6}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}$$

$$M = 1800$$

Summary

Combinatorics allows you to compute the odds of getting different results in Farkle.

- It's a little tricky, but the results are identical to enumeration
- And takes a lot less time