Pascal Distribution

ECE 341: Random Processes

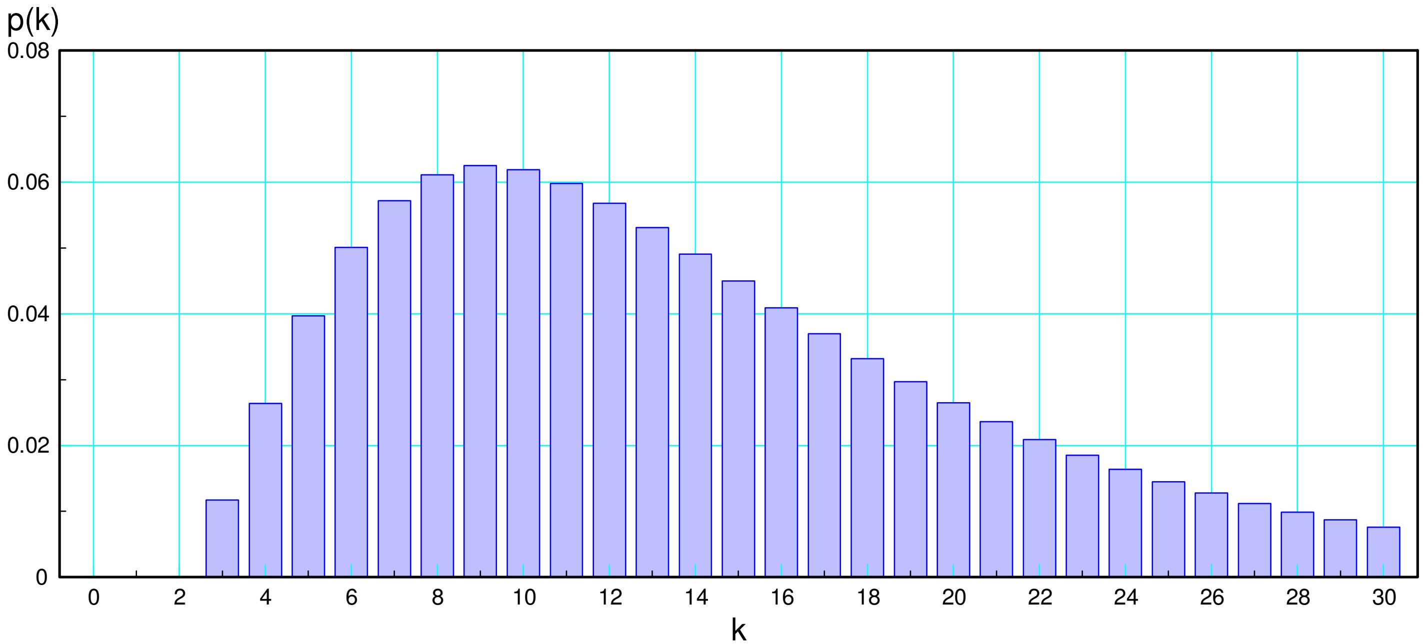
Lecture #11

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Pascal Distribution (a.k.a. Negative Binomial Distribution)

A Pascal distribution is similar to a geometric distribution, modeling

- The number of times you roll a die until you get r ones
- The number of times you make a trip with a car until r things fail (and it's time to buy a new car)
- The number of days until r accidents happen at work and you're promoted (Peter principle)



pdf - mgf - mean - variance

Distribution	description	pdf	mgf	mean	variance
Bernoulli trial	flip a coin obtain m heads	$p^m q^{1-m}$	$q+p/z$	p	p(1-p)
Binomial	flip n coins obtain m heads	$\binom{n}{m} p^m q^{n-m}$	$(q+p/z)^n$	np	np(1-p)
Hyper Geometric	Bernoulli trial without replacement	$\dfrac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$			
Uniform range = (a,b)	toss an n-sided die	$1/n \quad a \le m \le b$ $0 \quad otherwise$	$\left(\dfrac{1+z+z^2+\ldots+z^{n-1}}{n\, z^b}\right)$	$\left(\dfrac{a+b}{2}\right)$	$\left(\dfrac{(b+1-a)^2-1}{12}\right)$
Geometric	Bernoulli until 1st success	$p\, q^{k-1}$	$\left(\dfrac{p}{z-q}\right)$	$\left(\dfrac{1}{p}\right)$	$\left(\dfrac{q}{p^2}\right)$
Pascal	Bernoulli until rth success	$\binom{x-1}{r-1} p^r q^{x-r}$	$\left(\dfrac{p}{z-q}\right)^r$	$\left(\dfrac{r}{p}\right)$	$\left(\dfrac{qr}{p^2}\right)$

Source: Wikipedia

pdf for a Pascal Distribution

Assume you toss a coin with the probability of heads being p.

The probability of getting r heads on kth flip is:

- On the kth flip, you must get a heads (probability = p):
- On the previous k-1 flips, you got r-1 heads. (binomial distribution)

$$f(k) = \binom{k-1}{r-1} p^{r-1} q^{(k-1)-(r-1)}$$

- Both must happen

$$f(k) = p \cdot \binom{k-1}{r-1} p^{r-1} q^{(k-1)-(r-1)} = \binom{k-1}{r-1} p^{r} q^{k-r}$$

Mean and Variance

The moment generating function for a geometric distibution is

$$\psi(z) = \left(\frac{p}{z-q}\right)$$

The moment generating for r geometric distributions (a Pascal distribution) is

$$\psi(z) = \left(\frac{p}{z-q}\right)^r$$

Zeroth Moment

$$m_0 = \psi(z-1) = 1$$

$$m_0 = \left(\frac{p}{1-q}\right)^r = 1^r = 1$$

This is a valid moment generating function (probabities add to one)

1st Moment (mean)

$$m_1 = -\psi'(z = 1)$$

$$m_1 = -\frac{d}{dz}\left(\left(\frac{p}{z-q}\right)^r\right)_{z=1}$$

$$m_1 = -\left(\frac{-r \cdot p^r}{(z-q)^{r+1}}\right)_{z=1}$$

$$m_1 = \left(\left(\frac{r \cdot p^r}{(p)^{r+1}}\right)\right) = \left(\frac{r}{p}\right)$$

$$\mu = m_1 = \left(\frac{r}{p}\right)$$

Second Moment

$$m_2 = \psi''(z = 1)$$

$$m_1 = \frac{d^2}{dz^2}\left(\left(\frac{p}{z-q}\right)^r\right)_{z=1}$$

$$m_2 = \left(\frac{d}{dz}\left(\frac{-r \cdot p^r}{(z-q)^{r+1}}\right)\right)_{z=1}$$

$$m_2 = \left(\left(\frac{r(r+1)p^r}{(z-q)^{r+2}}\right)\right)_{z=1}$$

$$m_2 = \left(\left(\frac{r(r+1)p^r}{p^{r+2}}\right)\right)$$

$$m_2 = \left(\frac{r(r+1)}{p^2}\right)$$

Variance:

$$\sigma^2 = m_2 - m_1 - m_1^2$$

$$\sigma^2 = \left(\frac{r(r+1)}{p^2}\right) - \left(\frac{r}{p}\right) - \left(\frac{r}{p}\right)^2$$

$$\sigma^2 = \left(\frac{r^2+r-pr-r^2}{p^2}\right)$$

$$\sigma^2 = \left(\frac{r(1-p)}{p^2}\right)$$

$$\sigma^2 = \left(\frac{qr}{p^2}\right)$$

Example 1: Roll Until 1 Success

- Rolling a die until you get a 1 or 2
- The number of times you do the dishes until someone notices
- The number of parties you go to until you catch COVID-19

Solution: This is a geometric distribution

$$p(k) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-1} u(k-1)$$

Matlab Code:

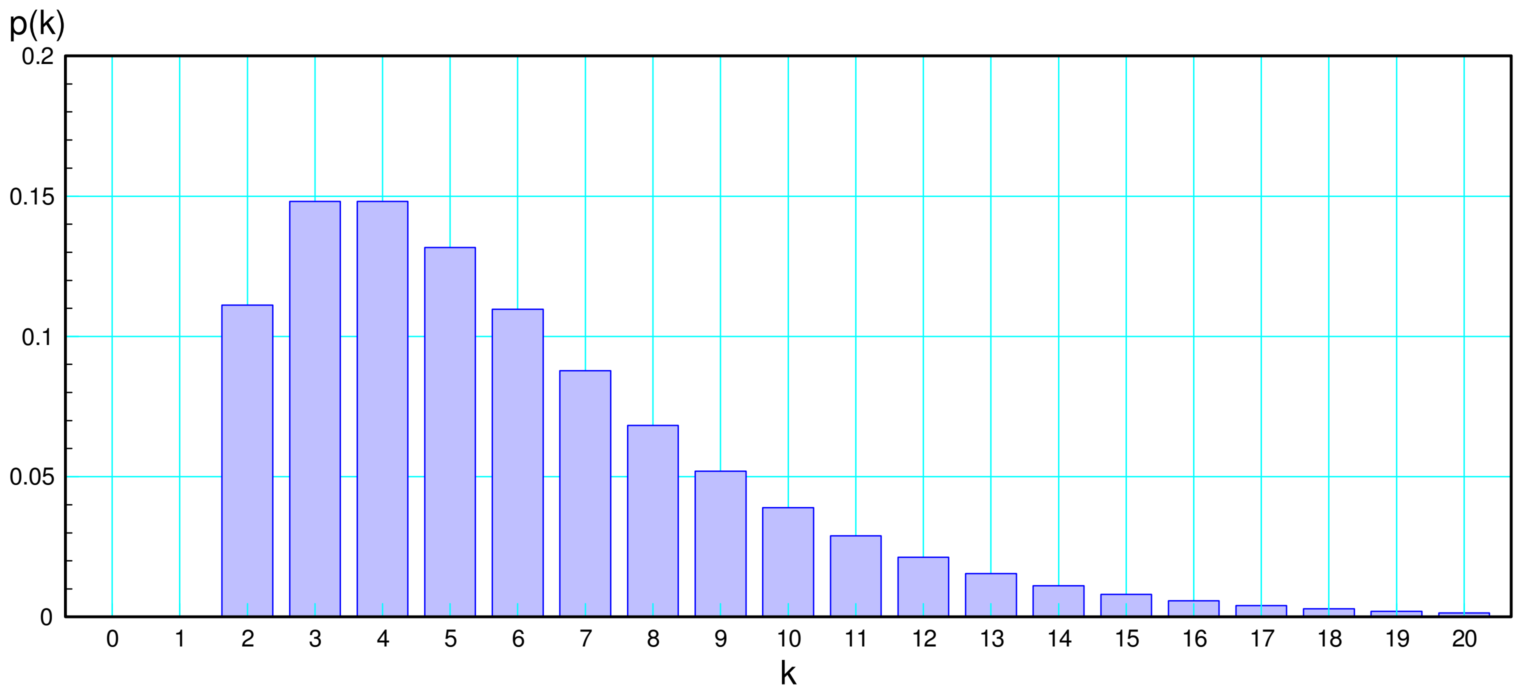
```
k = [1:30]';
A = (1/3) * (2/3).^(k-1) .* (k-1 >= 0);
k = [0;k];
A = [0;A];

bar(k,A)
```

Example 2: Roll until you get 2 successes

- The number of times you roll a 6-sided die until you roll a 1 or 2 twice (p = 1/3)
- The number of times you do the dishes until two people notice (p = 1/3)
- The number of parties you go to until you have two exposes to COVID-19 (assume p = 1/3)

Solution: This is a Pascal distribution.



Option 1: Convolution in matlab.

- We have the pdf for r = 1 from before
- Use convolution to repeat the event

```
A2 = conv(A,A);
[k(1:21), A2(1:21)]

        k          A2
           0           0
      1.0000           0
      2.0000      0.1111
      3.0000      0.1481
      4.0000      0.1481
      5.0000      0.1317
      6.0000      0.1097
      7.0000      0.0878
      8.0000      0.0683
      9.0000      0.0520
     10.0000      0.0390
     11.0000      0.0289
```

Option 2: Use the Pascal distribution formula

$$p(k) = \binom{k-1}{r-1} p^r q^{k-r}$$

```
p = 1/3;
q = 2/3;
r = 2;
B = zeros(21,1);
for i=3:20
   k = i-1;
   B(i) = NchooseM(k-1, r-1) * p^r * q^(k-r) .* (k >= 0);
   end

k = [0:20]';
[k(1:21),A2(1:21),B(1:21)]
```

k	conv	formula
0	0	0
1.0000	0	0
2.0000	0.1111	0.1111
3.0000	0.1481	0.1481
4.0000	0.1481	0.1481
5.0000	0.1317	0.1317
6.0000	0.1097	0.1097

Option 3: z-transforms

The moment-generating function (i.e. z-transform) for a geometric distribution is

$$\psi(z) = \left(\frac{p}{z-q}\right)$$

Doing this twice gives

$$\psi(z) = \left(\frac{p}{z-q}\right)^2 = \left(\frac{1/3}{z-2/3}\right)^2$$

Take the inverse z-transform. From a table of z-transforms:

$$\left(\frac{z}{(z-a)^2}\right) \leftrightarrow \left(\frac{k}{1!}\right) a^{k-1}$$

so

$$\left(\frac{1/3}{z-2/3}\right)^2 = \left(\frac{1}{9z}\right)\left(\frac{z}{(z-2/3)^2}\right) \rightarrow \left(\frac{1}{9z}\right)\ k \left(\frac{2}{3}\right)^{k-1}\ u(k)$$

1/z means delay by one

$$p(k) = \left(\frac{1}{9}\right) (k-1) \left(\frac{2}{3}\right)^{k-2} u(k-1)$$

Checking in Matlab

```
C = zeros(21,1);
for i=3:21
    k = i-1;
    C(i) = (1/9)*(k-1)* ( (2/3)^(k-2) );
    end

k = [0:20]';
[k(1:21),A2(1:21),B(1:21), C(1:21)]

    k      conv     formula    z-trans
    0          0          0          0
    1          0          0          0
    2     0.1111     0.1111     0.1111
    3     0.1481     0.1481     0.1481
    4     0.1481     0.1481     0.1481
    5     0.1317     0.1317     0.1317
    6     0.1097     0.1097     0.1097
    7     0.0878     0.0878     0.0878
```

Example 3: Combining Geometric Distributions

Assume you are rolling an 8-sided die

Let

- A be the number of rolls until you roll a 1 ($p = 1/8$)
- B be the number of rolls until you roll a 1 or 2 ($p = 2/8$)
- C be the number of rolls until you roll a 1, 2, or 3 ($p = 3/8$)

Determine the pdf for A + B + C

Solution using Matlab and Convolutions:

A, B, and C are geometric distributions

$$A(k) = \left(\frac{1}{8}\right)\left(\frac{7}{8}\right)^{k-1} u(k-1)$$

$$B(k) = \left(\frac{2}{8}\right)\left(\frac{6}{8}\right)^{k-1} u(k-1)$$

$$C(k) = \left(\frac{3}{8}\right)\left(\frac{5}{8}\right)^{k-1} u(k-1)$$

Use Matlab to convolve the three together

```
A = (1/8) * (7/8).^(k-1) .* (k-1 >= 0);
B = (2/8) * (6/8).^(k-1) .* (k-1 >= 0);
C = (3/8) * (5/8).^(k-1) .* (k-1 >= 0);
```

k	A	B	C
0	0	0	0
1.0000	0.1250	0.2500	0.3750
2.0000	0.1094	0.1875	0.2344
3.0000	0.0957	0.1406	0.1465
4.0000	0.0837	0.1055	0.0916

Now convolve them

```
AB = conv(A,B);
ABC = conv(AB,C);
[k(1:21),ABC(1:21)]
```

k	p(k)
0	0
1	0
2	0
3	0.0117
4	0.0264
5	0.0397
6	0.0501
7	0.0572
8	0.0611
9	0.0625
10	0.0619

Option 2: z-transforms

$$A(z) = \left(\frac{1/8}{z-7/8}\right)$$

$$B(z) = \left(\frac{2/8}{z-6/8}\right)$$

$$C(z) = \left(\frac{3/8}{z-5/8}\right)$$

The z-transform for the sum of the three is

$$Y(z) = A(z) \cdot B(z) \cdot C(z)$$

$$Y(z) = \left(\frac{1/8}{z-7/8}\right)\left(\frac{2/8}{z-6/8}\right)\left(\frac{3/8}{z-5/8}\right)$$

This isn't in the table of z-transforms, so use partial fraction expansion

$$Y(z) = \left(\frac{1/8}{z-7/8}\right)\left(\frac{2/8}{z-6/8}\right)\left(\frac{3/8}{z-5/8}\right)$$

$$Y(z) = \left(\frac{0.375}{z-7/8}\right) + \left(\frac{-0.75}{z-6/8}\right) + \left(\frac{0.375}{z-5/8}\right)$$

Multiply both sides by z

$$zY = \left(\frac{0.375z}{z-7/8}\right) + \left(\frac{-0.75z}{z-6/8}\right) + \left(\frac{0.375z}{z-5/8}\right)$$

$$z \cdot y(k) = \left(0.375\left(\frac{7}{8}\right)^{k} - 0.75\left(\frac{6}{8}\right)^{k} + 0.375\left(\frac{5}{8}\right)^{k}\right)u(k)$$

Divide by z (delay by 1)

$$y(k) = \left(0.375\left(\frac{7}{8}\right)^{k-1} - 0.75\left(\frac{6}{8}\right)^{k-1} + 0.375\left(\frac{5}{8}\right)^{k-1}\right)u(k-1)$$

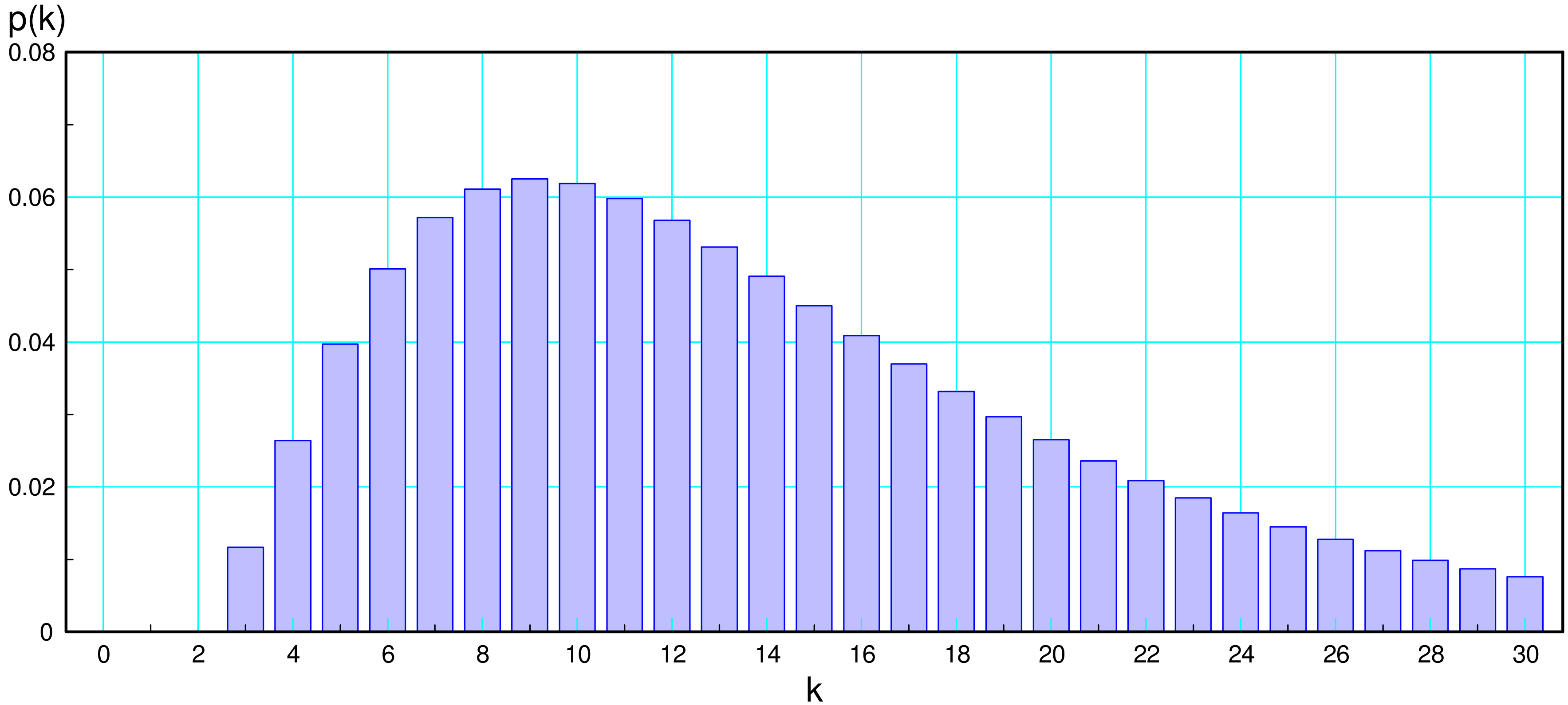
Checking against the results for convolution:

```
k = [0:31]';
Y = 0*k;
Y = ( 0.375*(7/8).^(k-1) - 0.75*(6/8).^(k-1) + 0.375*(5/8).^(k-1) ) .*
(k-1>0);

[k(1:21),ABC(1:21),Y(1:21)]
```

k	conv	z-trans
0	0	0
1	0	0
2	0	0
3	0.0117	0.0117
4	0.0264	0.0264
5	0.0397	0.0397
6	0.0501	0.0501
7	0.0572	0.0572
8	0.0611	0.0611
9	0.0625	0.0625
10	0.0619	0.0619
11	0.0598	0.0598

Either method is valid: they give you the same results.



pdf for the number of rolls of an 8-sided die until you roll a 1, then a 1 or 2, then a 1, 2, or 3

Example 3:

Determine the pdf for X where X is the total number of die rolls:

- Roll an 8-sided die until you get four ones, then
- Roll a 6-sided die until you get two ones.

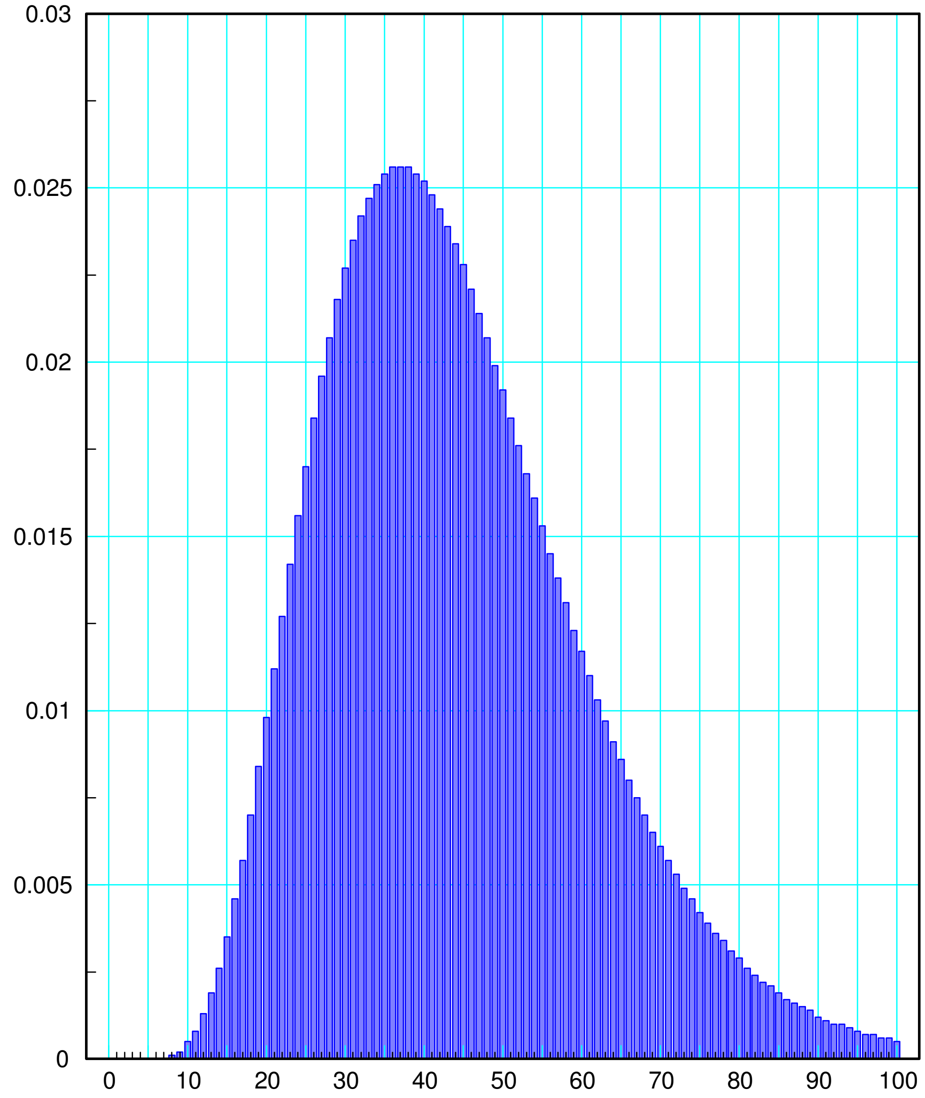
Matlab Code

```
>> k = [0:100]';
>> d8 = 1/8 * (7/8).^(k-1);
>> d8(1) = 0;
>> d6 = 1/6 * (5/6).^(k-1);
>> d6(1) = 0;
>> d8x2 = conv(d8,d8);
>> d8x4 = conv(d8x2,d8x2);
>> d6x2 = conv(d6,d6);
>> X = conv(d8x4,d6x2);
>> sum(X)

ans =     1.0000

>> X = X(1:101);
>> bar(k,X)
```

pdf



Find an equation for the pdf:

- z-transforms helps here

The number of die rolls until you get a one on a 6 and 8-sided die:

$$d6 = \left(\frac{1/6}{z-5/6}\right) \qquad d8 = \left(\frac{1/8}{z-7/8}\right)$$

The number of die rolls until you four 1's on a d8 and two 1's on d6

$$X = \left(\frac{1/6}{z-5/6}\right)^2 \left(\frac{1/8}{z-7/8}\right)^4$$

Now take the inverse z-transform

- partial fractions is going to be a pain...

Engineering Solution

- Change the problem so that it's easier to solve
- While keeping the flavor of the actual problem

Change X

$$X = \left(\frac{1/6}{z-0.8333}\right)^2 \left(\frac{1/8}{z-0.8750}\right)^4$$

to

$$X \approx \left(\frac{-}{z-0.8323}\right)\left(\frac{-}{z-0.8343}\right)\left(\frac{-}{z-0.873}\right)\left(\frac{-}{z-0.874}\right)\left(\frac{-}{z-0.876}\right)\left(\frac{-}{z-0.877}\right)$$

Adjust the numerator so the DC gain is 1.000

- the zeroth moment is 1.000
- this is a valid PDF

$$X = \left(\frac{0.1677}{z-0.8323}\right)\left(\frac{0.1657}{z-0.8343}\right)\left(\frac{0.127}{z-0.873}\right)\left(\frac{0.126}{z-0.874}\right)\left(\frac{0.124}{z-0.876}\right)\left(\frac{0.123}{z-0.877}\right)$$

This has a delay of six, so multuply by z^6

$$z^6 X = \left(\frac{0.1677z}{z-0.8323}\right)\left(\frac{0.1657z}{z-0.8343}\right)\left(\frac{0.127z}{z-0.873}\right)\left(\frac{0.126z}{z-0.874}\right)\left(\frac{0.124z}{z-0.876}\right)\left(\frac{0.123z}{z-0.877}\right)$$

Now do partial fractions (keep lots of decimal places)

$$z^6 X = \left(\frac{az}{z-0.8323}\right) + \left(\frac{bz}{z-0.8343}\right) + \left(\frac{cz}{z-0.873}\right) + \left(\frac{dz}{z-0.874}\right) + \left(\frac{ez}{z-0.876}\right) + \left(\frac{fz}{z-0.877}\right)$$

Take the inverse z-transform

$$x(k+6) = a(0.8323)^k + b(0.8343)^k + c(0.873)^k + d(0.874)^k + e(0.876)^k + f(0.877)^k$$

The results is almost the same

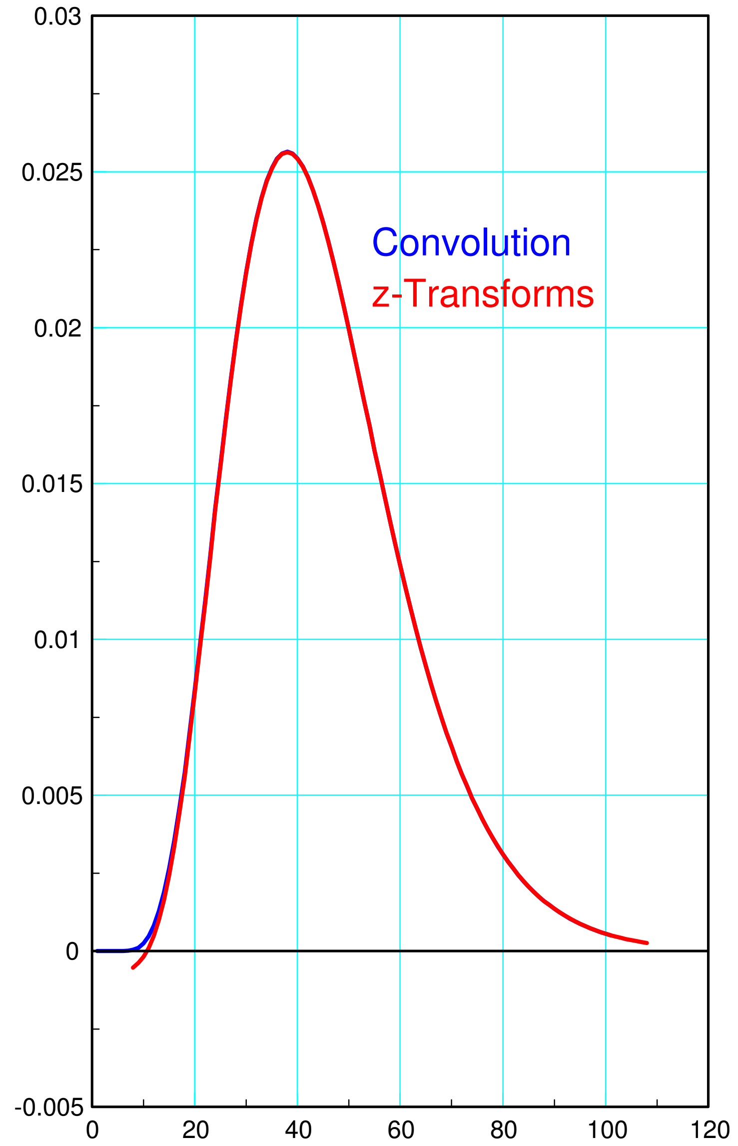
```
poles = [0.8323,0.8343,0.873,0.874,0.876,0.877];
G = zpk([0,0,0,0,0],poles,1,1);
DC = evalfr(G,1);
k = 1/DC;
G = zpk([0,0,0,0,0],poles,k,1);

eps = 1e-12'

z = 0.8323 + eps;
a = evalfr(G,z) * (z-0.8323);
z = 0.8343 + eps;
b = evalfr(G,z) * (z-0.8343);
z = 0.873 + eps;
c = evalfr(G,z) * (z-0.873);
z = 0.874 + eps;
d = evalfr(G,z) * (z-0.874);
z = 0.876 + eps;
e = evalfr(G,z) * (z-0.876);
z = 0.877 + eps;
f = evalfr(G,z) * (z-0.877);

k = [1:101]';
Y = a*0.8323.^k + b*0.8343.^k + c*0.873.^k +
d*0.874.^k + e*0.876.^k + f*0.877.^k;

plot(k,X,k+7,Y)
```



Summary

A Pascal distribution is an extesion of a geometric distribution

- The number of rolls until k event happen

This is where z-transforms shine

- They make computing the mean and variance easier
- They make combining geometric distributions easier