Uniform Distribution

ECE 341: Random Processes

Lecture #14

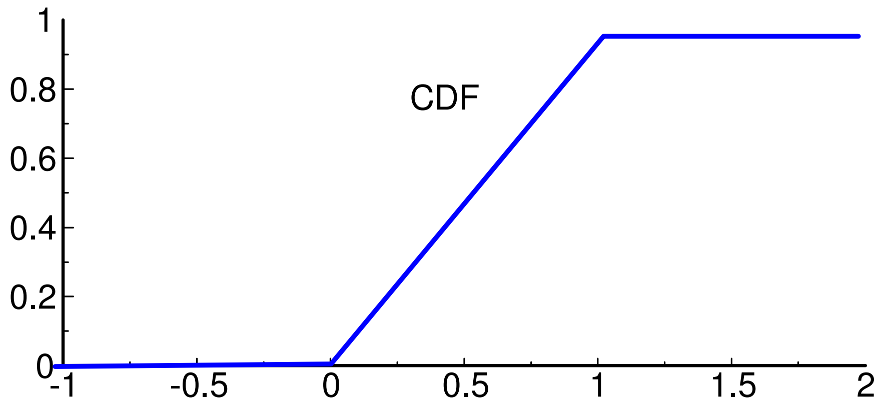
note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

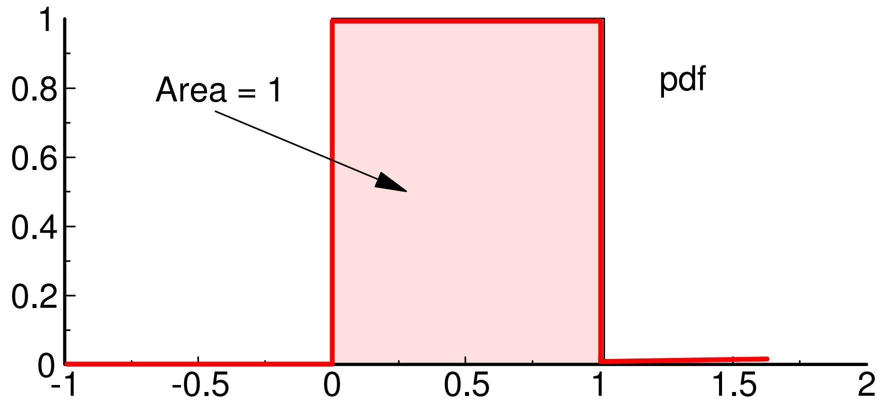
Uniform Distribution

A uniform distribution is one where the probability density function (pdf) is

- constant over a range (a, b), and
- zero otherwise.

Example: Uniform distribution over the interval of (1, 2):

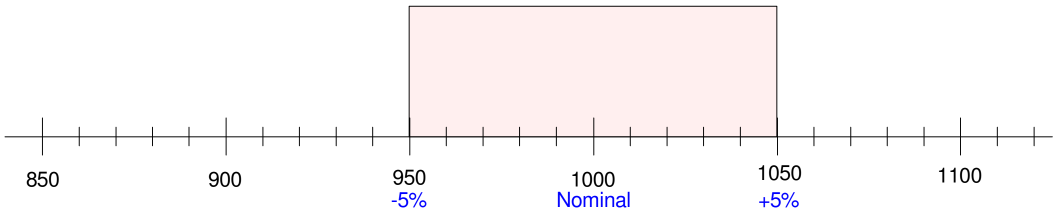




Examples:

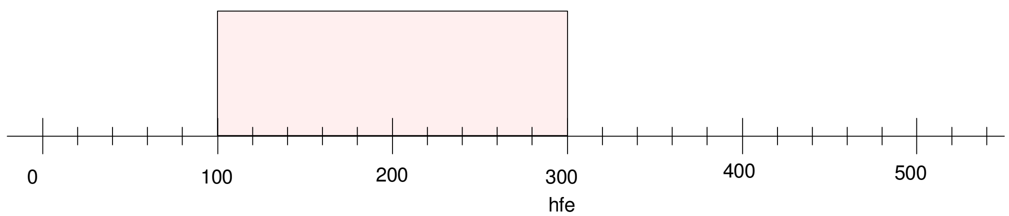
A 1k resistor

- 5% tolerance
- R = 1k +/- 5%



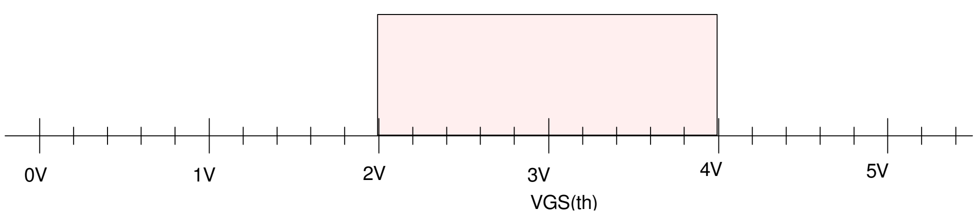
BJT Transistor

- min(hfe) = 100
- max(hfe) = 300



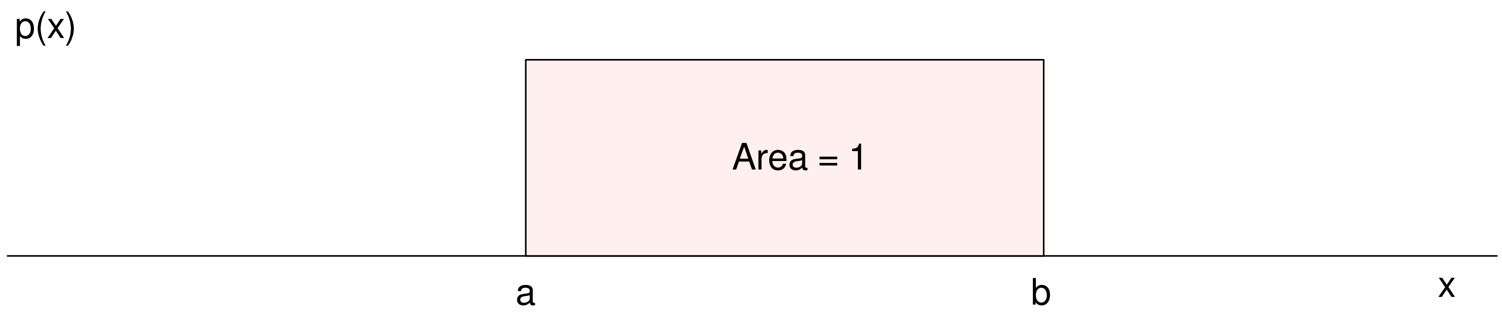
MOSFET

- $\min(V_{GS(th)}) = 2V$
- $\max(V_{GS(th)}) = 4V$



Properties of Uniform Distributions:

Assume a uniform distribution over the interval (a, b)



Since the area must be 1.000, the height must be

Area = width * height = 1

$p(x) = \left(\frac{1}{b-a}\right)$ $\quad$ a < x < b

Mean: The mean of the funciton (almost by inspection) is

$$\bar{x} = \left(\frac{a+b}{2}\right)$$

Variance:

$$\sigma^2 = \int_a^b p(x)\ (x-\mu)^2\ dx$$

$$\sigma^2 = \left(\frac{(b-a)^2}{12}\right)$$

Moment Generating Function

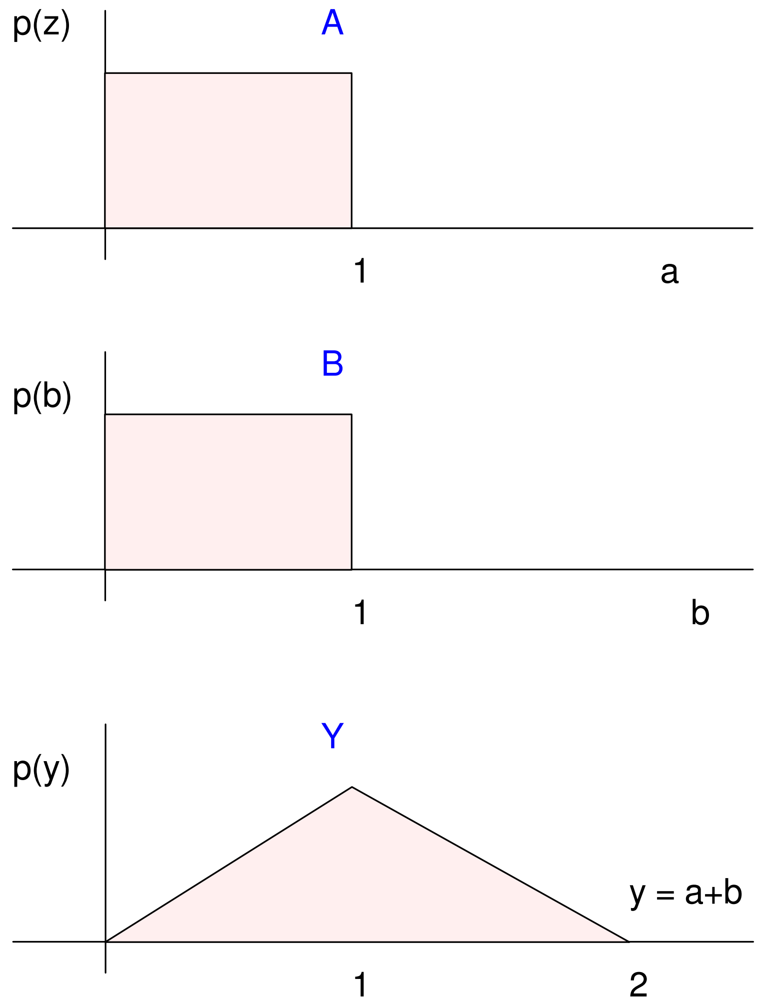
$$\psi(s) = \left(\frac{1}{(b-a)s}\right)(e^{-as} - e^{-bs})$$

Combinations of Uniform Distributions

- Net resistance of two resistors in series

Adding two uniform distributions, the results in

- The colvolution of their pdf's, or
- The product of their moment generating functions.



Example:

- Assume A and B are uniform distributions over the interval (0, 1).
- Find the pdf of the sum Y = A + B.

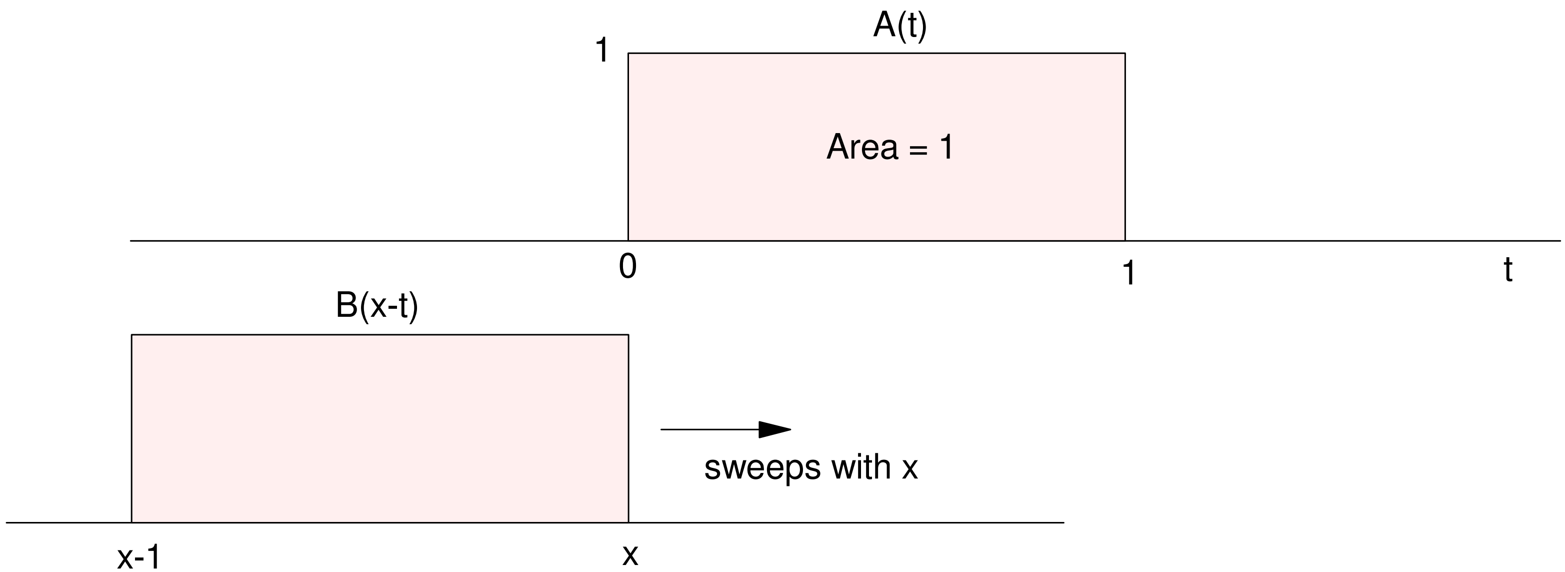
Solution using convolution:

$$A(x) = u(x) - u(x-1)$$

$$B(x) = u(x) - u(x-1)$$

$$Y(x) = A(x) ** B(x)$$

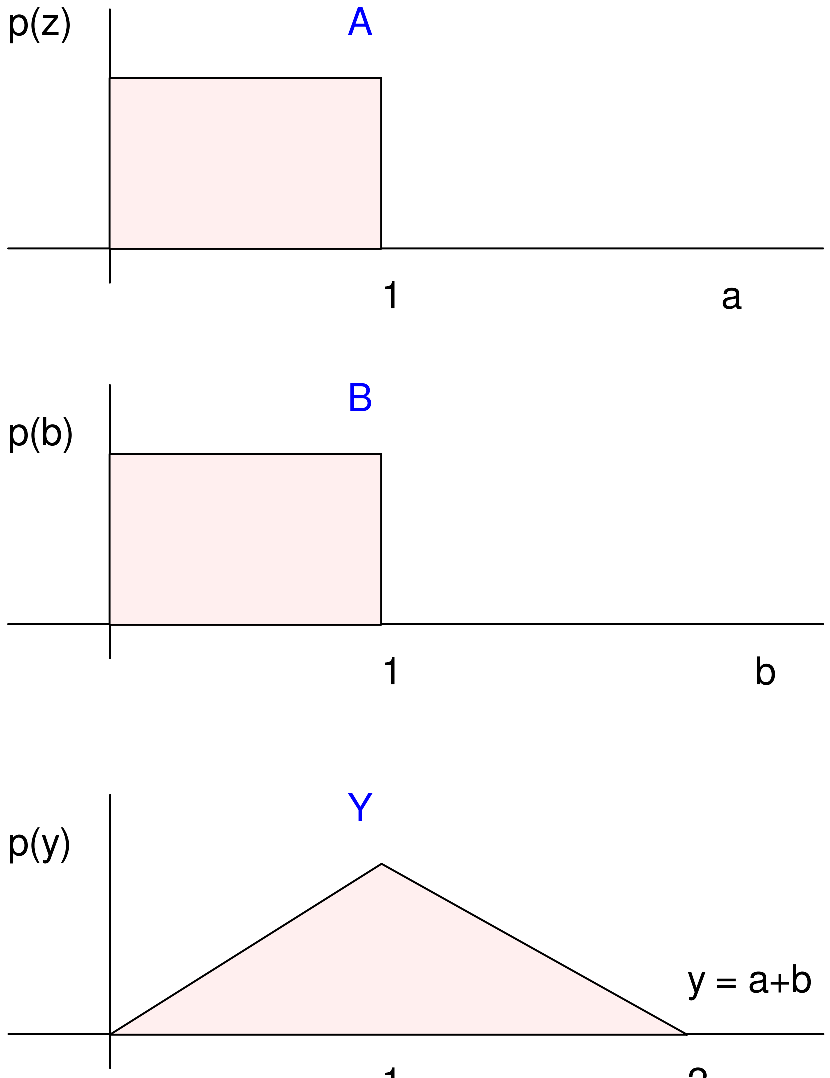
$$Y(x) = \int_{-\infty}^{\infty} A(t)B(x-t)dt$$



From the graph, this integral is

Y(x) =

- 0 $\qquad x < 0$
- x $\qquad 0 < x < 1$
- 1 - x $\qquad 1 < x < 2$
- 0 $\qquad 2 < x$



Solution using Calculus:

$$Y(x) = \int_{-\infty}^{\infty} A(t)\ B(x-t)\ dt$$

$$Y(x) = \int_{-\infty}^{\infty} (u(t) - u(t-1))\ (u(x-t) - u(x-t-1))\ dt$$

Multiply out giving four terms

$$= \int_{-\infty}^{\infty} u(t)u(x-t)\ dt + \int_{-\infty}^{\infty} -u(t)u(x-t-1))\ dt$$

$$+ \int_{-\infty}^{\infty} -u(t-1)u(x-t))\ dt + \int_{-\infty}^{\infty} u(t-1)\ u(x-t-1))\ dt$$

$$= \int_{0}^{x} 1\ dt - \int_{0}^{x-1} 1\ dt - \int_{1}^{x} 1\ dt + \int_{1}^{x-1} 1 dt$$

$$= xu(x) - (x-1)u(x-1) - (x-1)u(x-1) + (x-2)u(x-2)$$

$$= xu(x) - 2(x-1)u(x-1) + (x-2)u(x-2)$$

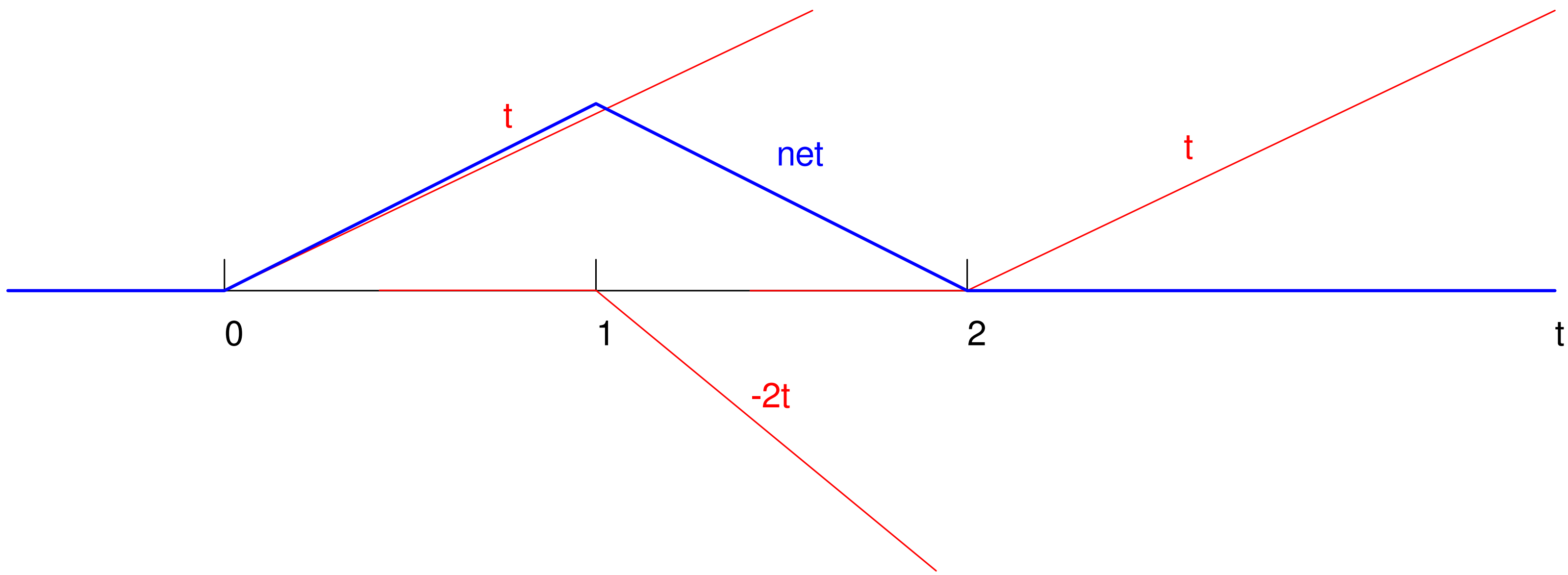
Solve using moment generating functions

$$A(s) = \left(\frac{1}{s}\right)(1 - e^{-s}) \qquad B(s) = \left(\frac{1}{s}\right)(1 - e^{-s})$$

$$Y(s) = A(s)B(s) \qquad Y(s) = \left(\frac{1}{s^2}\right)(e^{-2s} - 2e^{-s} + 1)$$

Take the inverse LaPlace transform

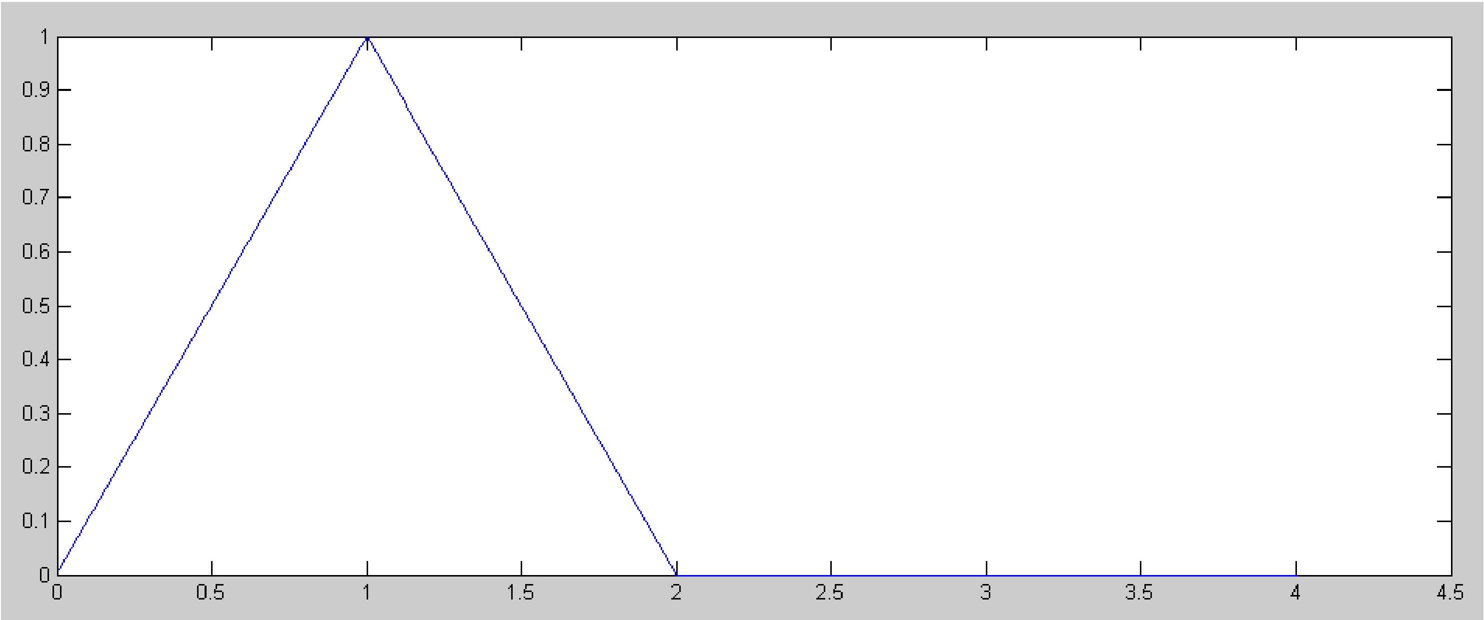
$$y(x) = (x-2)\ u(x-2) - 2(x-1)\ u(x-1) + x\ u(x)$$



Solve in Matlab

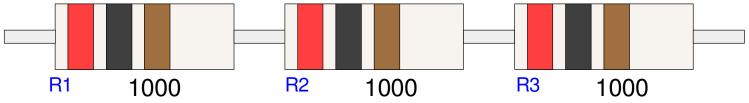
- Approximate a uniform distribution with 100 points over the interval (0, 1)

```
dx = 0.01;
x = [0:dx:2]';;
A = 1*(x<1);
B = 1*(x<1);
Y = conv(A, B) * dx;
plot([1:length(Y)]*dx,Y)
```



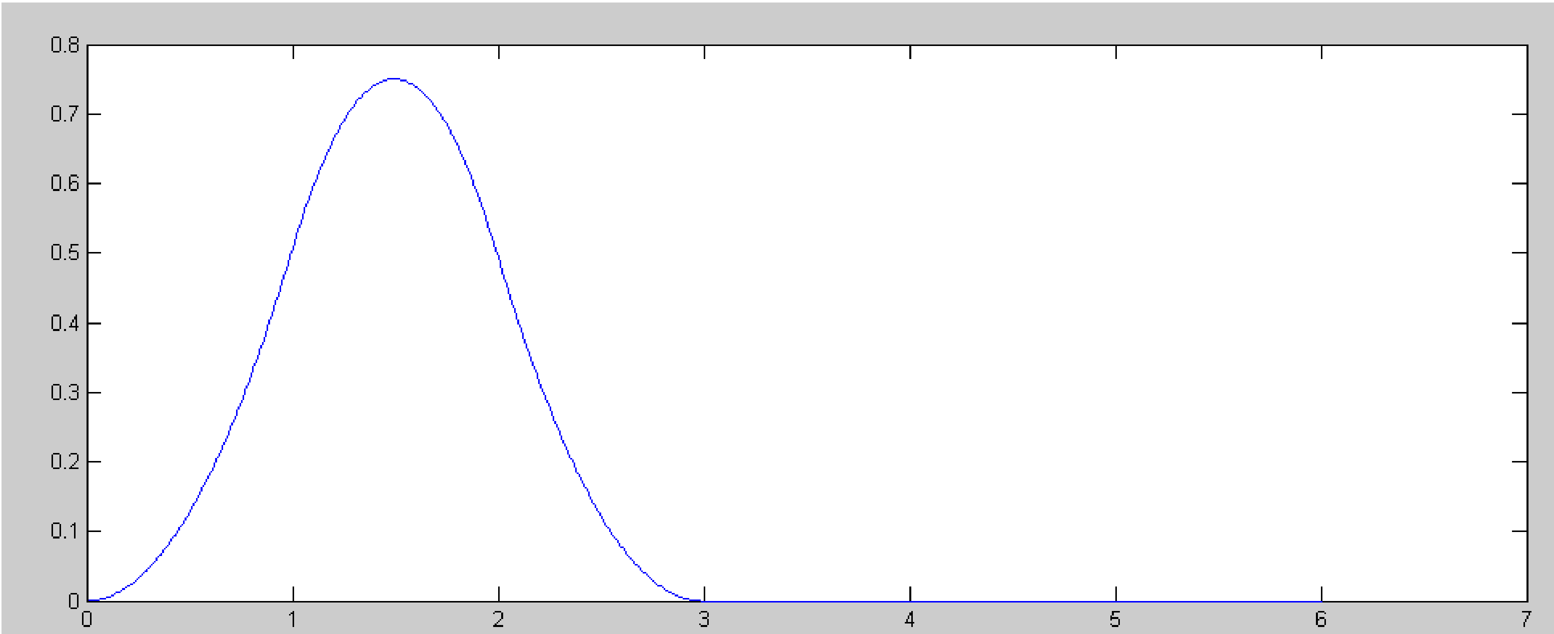
Example 2: Three uniform distribitions

- Example: Three resistors in series



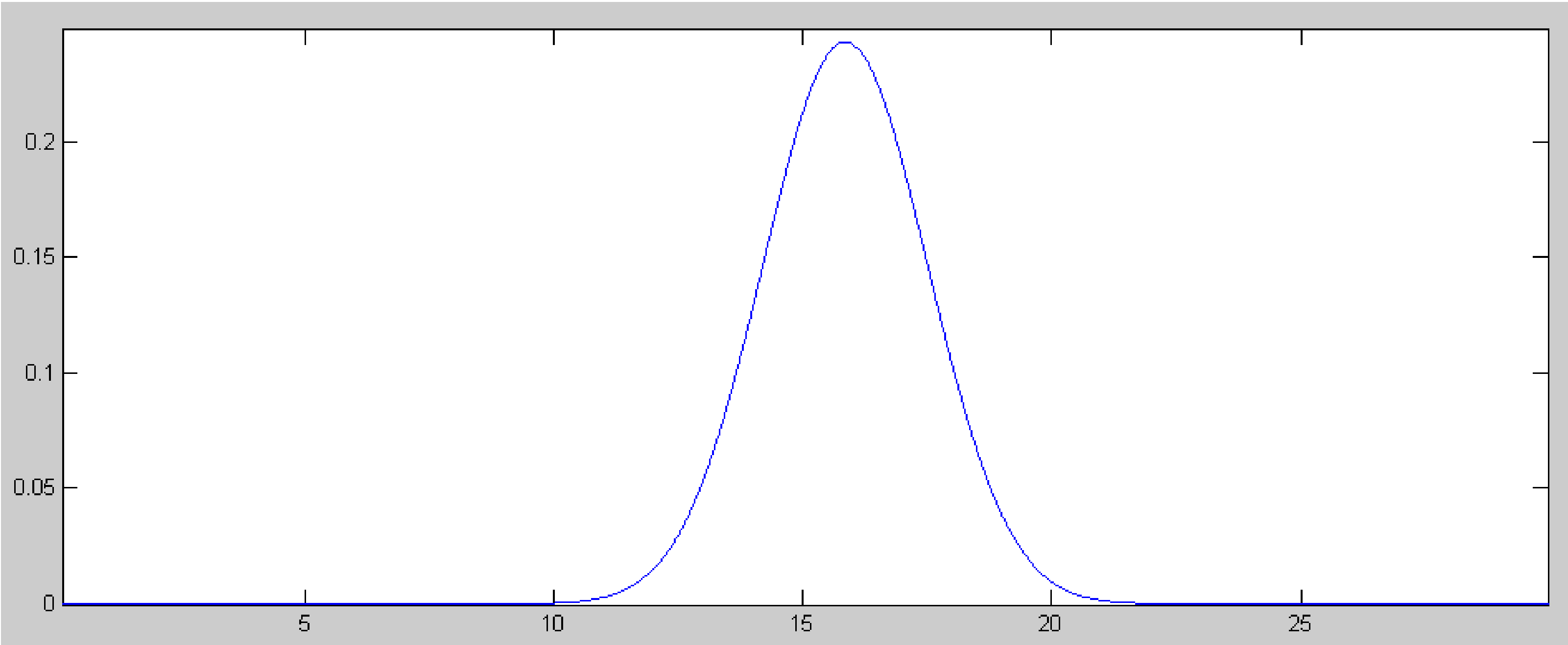

```
x = [0:dx:2]';
A = 1*(x<1);
B = 1*(x<1);
C = 1*(x<1);

Y = conv(A, B) * dx;
Y = conv(Y, C) * dx;
plot([1:length(Y)]*dx,Y)
```



Example 3: Find the pdf for the sum of 32 uniform distributions:

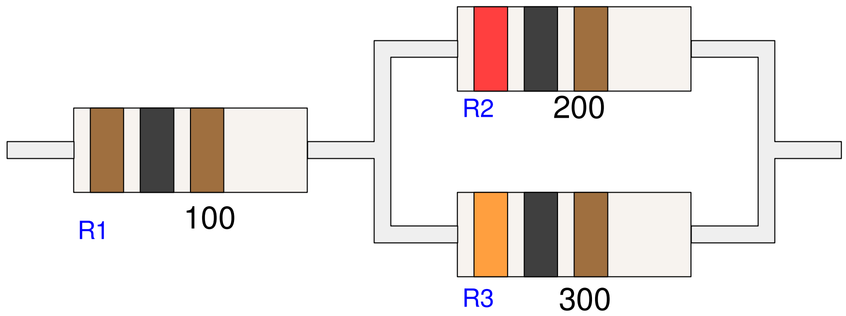
```
x = [0:dx:2]';
A = 1*(x<1);
Y2 = conv(A, A) * dx;
Y4 = conv(Y2, Y2) * dx;
Y8 = conv(Y4, Y4) * dx;
Y16 = conv(Y8, Y8) * dx;
Y32 = conv(Y16, Y16) * dx;
plot([1:length(Y32)]*dx,Y32)
```



Uniform Distribution in Circuit Analysis:

Find the pdf for the net resistance

- Each resistor has 5% tolerance



From Circuits I

$$R = R_1 + R_2 || R_3$$

$$R = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_3} \right)^{-1}$$

Problem:

- How do you combine uniform pdf's when you multiply?
- When you divide?

Monte-Carlo to the Rescue!

Monte-Carlo always works

cdf:

- Generate random values for R1, R2, R3
  - Uniform distribution
  - +/- 5%
- Solve for Rnet
- Repeat N tims

When done,

- Sort the data
- Plot the sorted data

The result is the cdf

```
Data = [];

for i=1:1e4
   R1 = 17600 * (1 + (rand()*2-1)*0.05);
   R2 = 2256 * (1 + (rand()*2-1)*0.05);
   Rc = 1000 * (1 + (rand()*2-1)*0.05);
   Re = 100 * (1 + (rand()*2-1)*0.05);
   Beta = 200 + 100*(rand()*2-1);
   Vb = 12*(R2 / (R1+R2));
   Rb = 1/(1/R1 + 1/R2);
   Ib = (Vb-0.7) / (Rb + (1+Beta)*Re);
   Ic = Beta*Ib;
   Vce = 12 - Rc*Ic - Re*(Ic+Ib);
   Data = [Data; Vce];
end

Data = sort(Data)
p = [1:length(Data)]' / length(Data);
plot(Data, p)
xlabel('Ohms')
```

Monte-Carlo Result

- cdf
- cdf's all look alike to me...

```
Data = [];

for i=1:1e4
   R1 = 17600 * (1 + (rand()*2-1)*0.05);
   R2 = 2256 * (1 + (rand()*2-1)*0.05);
   Rc = 1000 * (1 + (rand()*2-1)*0.05);
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Data = sort(Data)
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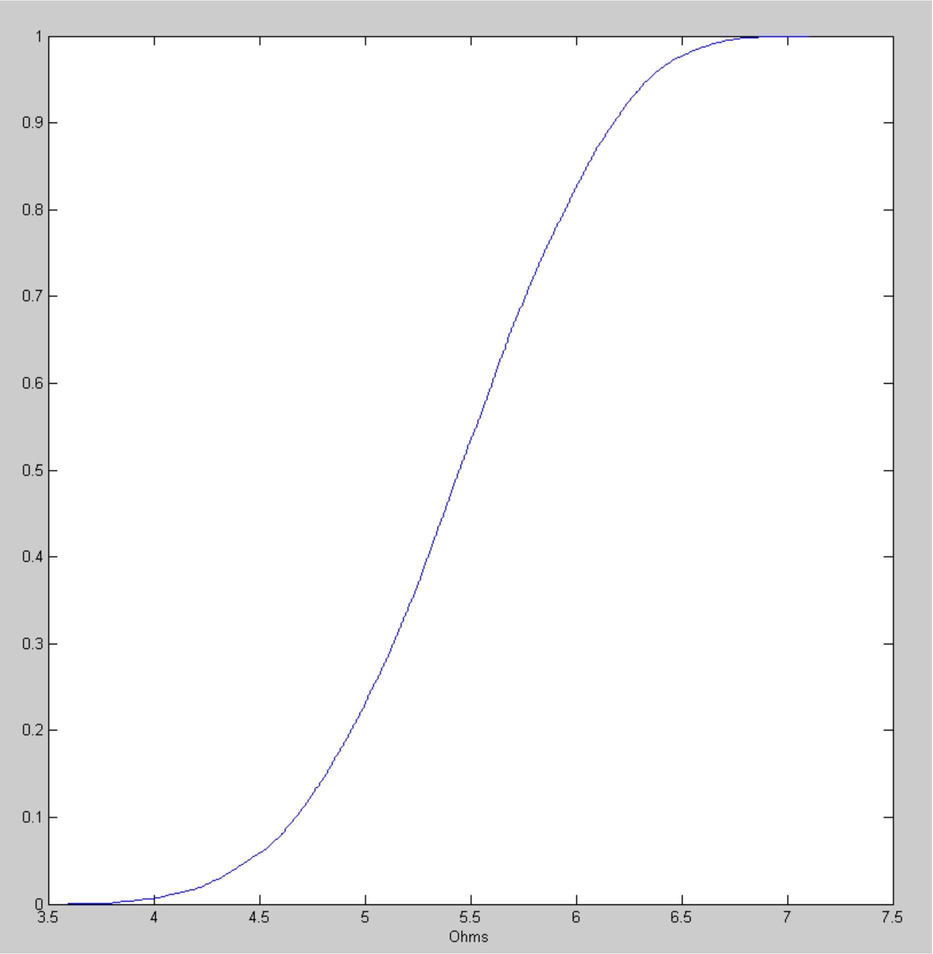
```



Monte Carlo for pdf

A little trickier

Create N bins to save the results

- Start from 300 Ohms
- Each bin is 0.1 Ohm wide

Compute 1 million values of Rnet

- Place in each bin

Resulting frequency of bins is pdf

```
X = zeros(2000,1);
Rmin = 200;
for i=1:1e6
   R1 = 190 * (1 + (rand()*2-1)*0.05);
   R2 = 200 * (1 + (rand()*2-1)*0.05);
   R3 = 300 * (1 + (rand()*2-1)*0.05);
   Rnet = R1 + 1 / (1/R2 + 1/R3);
   Bin = round((Rnet-Rmin)*10);
   X(Bin) = X(Bin) + 1;
end
X = X / 1e6;

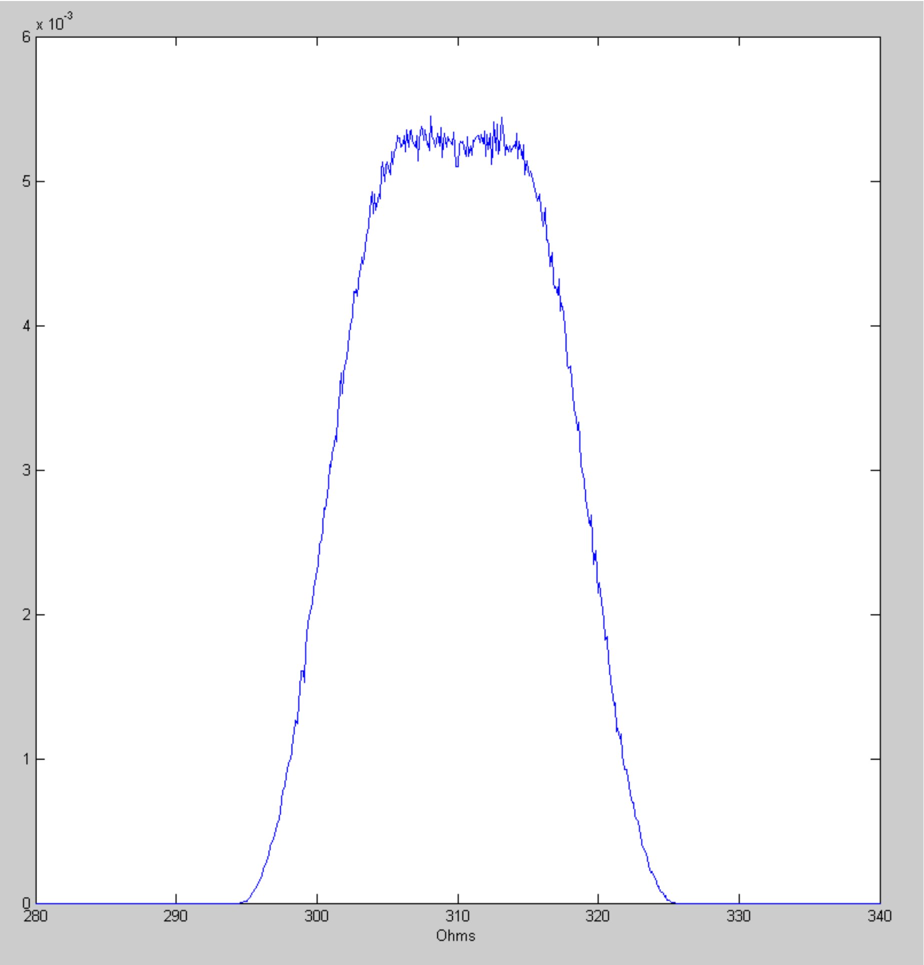
plot([1:2000]/10 + Rmin, X);
xlabel('Ohms')
```

Resulting pdf

Easier to see what the function looks like

```
X = zeros(2000,1);
Rmin = 200;
for i=1:1e6
   R1 = 190 * (1 + (rand()*2-1)*0.05);
   R2 = 200 * (1 + (rand()*2-1)*0.05);
   R3 = 300 * (1 + (rand()*2-1)*0.05);
   Rnet = R1 + 1 / (1/R2 + 1/R3);
   Bin = round((Rnet-Rmin)*10);
   X(Bin) = X(Bin) + 1;
end
X = X / 1e6;

plot([1:2000]/10 + Rmin, X);
xlabel('Ohms')
```

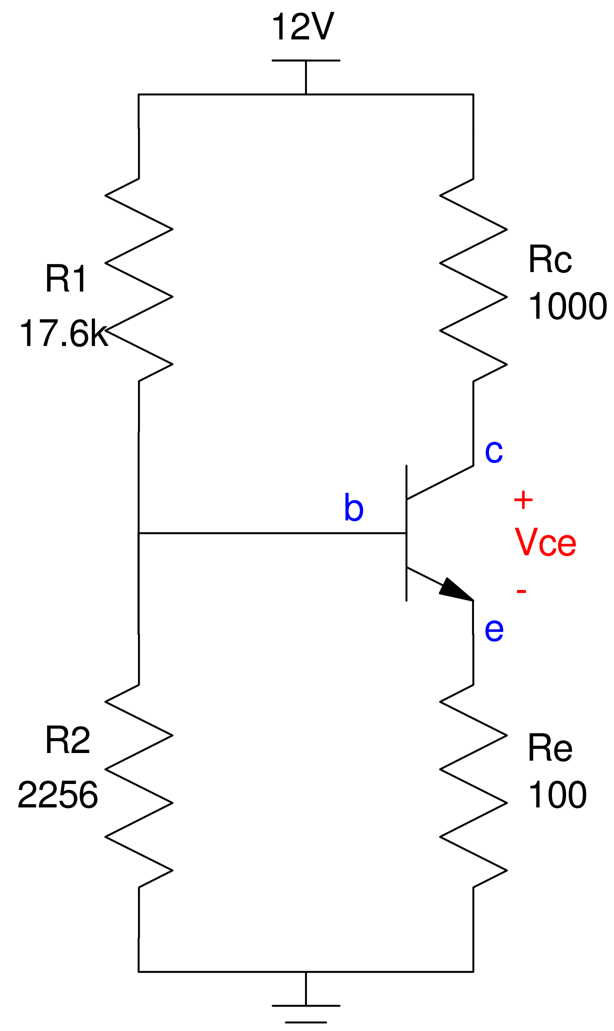


Example 2: BJT Circuit

- Circuit from Electronics

What is Vce assuming

- Each resistor has 5% tolerance
- $100 < \beta < 300$?



Circuit Equations:

- From Electronics

The equations for this circuit

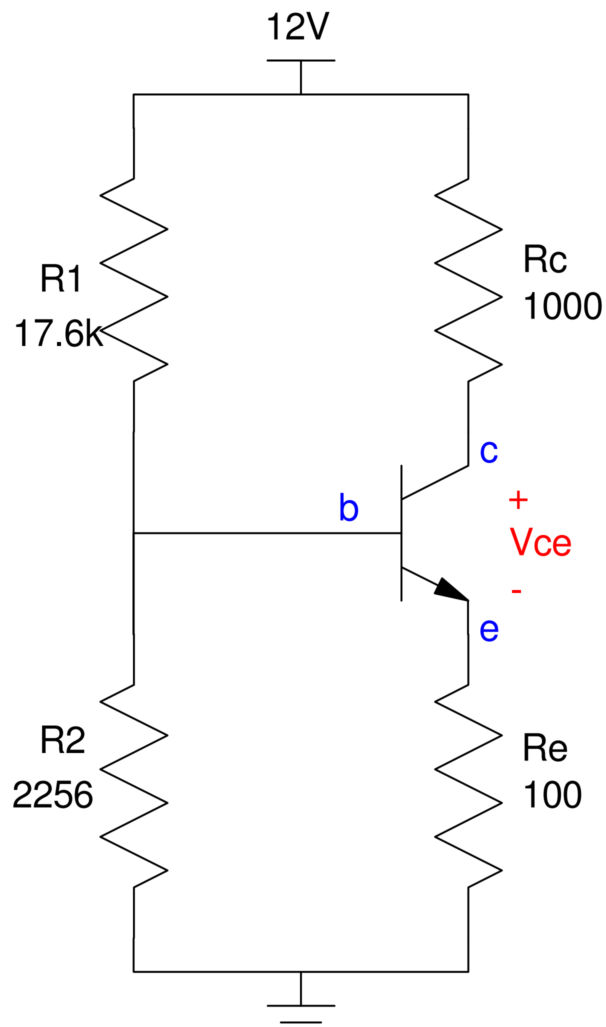
$$V_{th} = \left(\frac{R_2}{R_2+R_1}\right)12V$$

$$R_{th} = \left(\frac{R_1 R_2}{R_1+R_2}\right)$$

$$I_b = \left(\frac{V_{th}-.07}{R_{th}+(1+\beta)R_e}\right)$$

$$I_c = \beta I_b$$

$$V_{ce} = 12 - R_c I_c - R_e(I_b + I_c)$$



cdf from Monte-Carlo

- cdf's all look alike to me...

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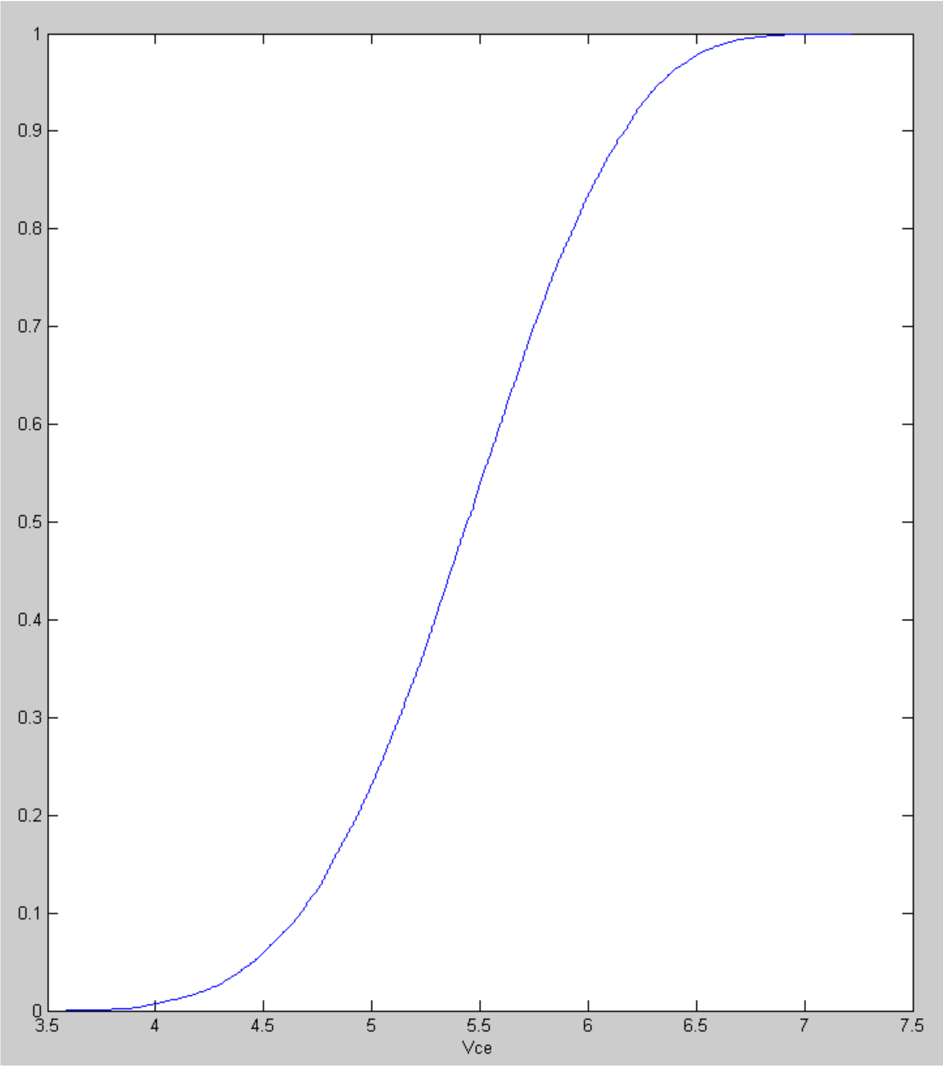
DATA = [];

for i=1:1e4
   R1 = 17600 * (1 + (rand()*2-1)*0.05);
   R2 = 2256 * (1 + (rand()*2-1)*0.05);
   Rc = 1000 * (1 + (rand()*2-1)*0.05);
   Re = 100 * (1 + (rand()*2-1)*0.05);
   Beta = 200 + 100*(rand()*2-1);
   Vb = 12*(R2 / (R1+R2));
   Rb = 1/(1/R1 + 1/R2);
   Ib = (Vb-0.7) / (Rb + (1+Beta)*Re);
   Ic = Beta*Ib;
   Vce = 12 - Rc*Ic - Re*(Ic+Ib);
   DATA = [DATA; Vce];
end

DATA = sort(DATA);
p = [1:length(DATA)]' / length(DATA);
plot(DATA, p)
xlabel('Vce')
```



pdf from Monte Carlo

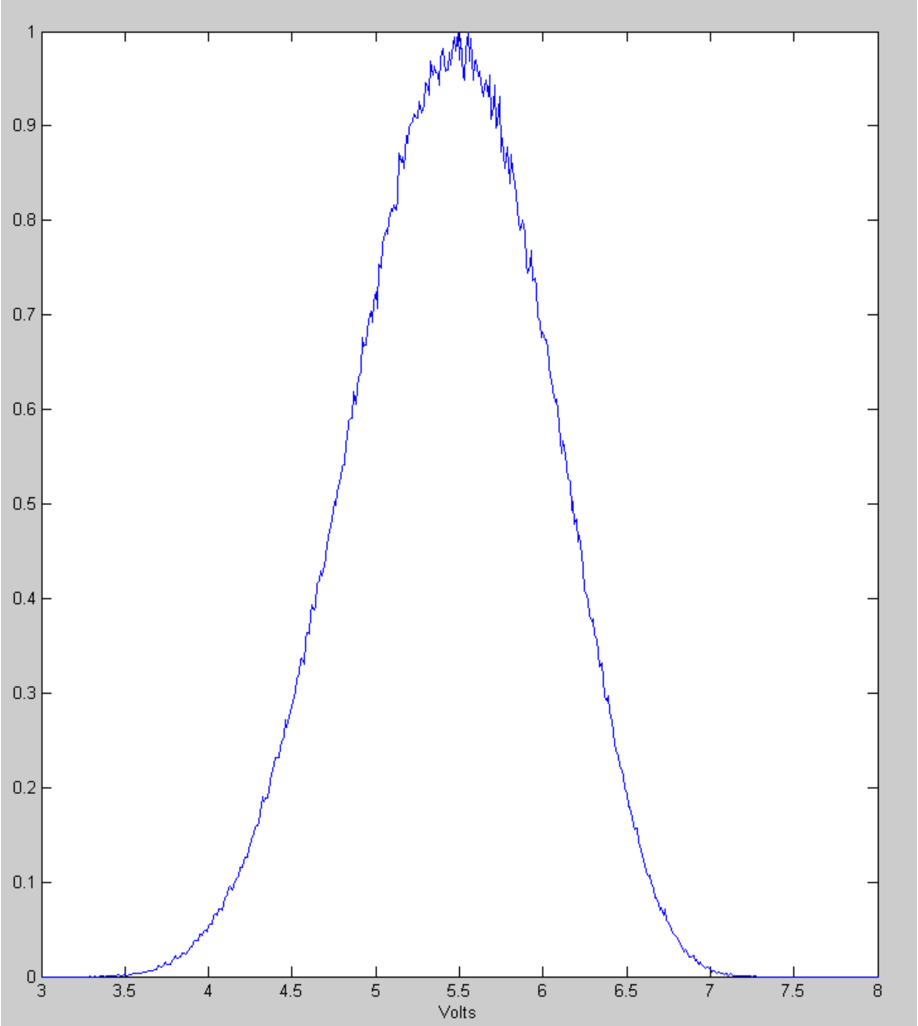
- pdf is more informative

```
Data = zeros(1000,1);

for i=1:1e6
   R1 = 17600 * (1 + (rand()*2-1)*0.05);
   R2 = 2256 * (1 + (rand()*2-1)*0.05);
   Rc = 1000 * (1 + (rand()*2-1)*0.05);
   Re = 100 * (1 + (rand()*2-1)*0.05);
   Beta = 200 + 100*(rand()*2-1);
   Vb = 12*(R2 / (R1+R2));
   Rb = 1/(1/R1 + 1/R2);
   Ib = (Vb-0.7) / (Rb + (1+Beta)*Re);
   Ic = Beta*Ib;
   Vce = 12 - Rc*Ic - Re*(Ic+Ib);
   Bin = round(Vce*100);
   Data(Bin) = Data(Bin) + 1;
end

Data = Data / max(Data);

V = [1:1000]'/100;
plot(V, Data);
xlabel('Volts');
```



Summary

Uniform distrubutions are fairly common and easy to model

- rand() function in Matlab

The sum of uniform distributions converges to a Normal distribution

- Central Limit Theorem (coming soon)

Convolution is required to sum uniform distributions

- conv() in Matlab
- Or you can use multipliation if using LaPlace transforms
  - a.k.a. moment generating functions in statistics

Monte-Carlo is useful when combining uniform distributions other ways

- Circuit analysis