# Normal Distribution

## a.k.a. Gaussian Distribution

# ECE 341: Random Processes

## Lecture #15


North Dakota State University

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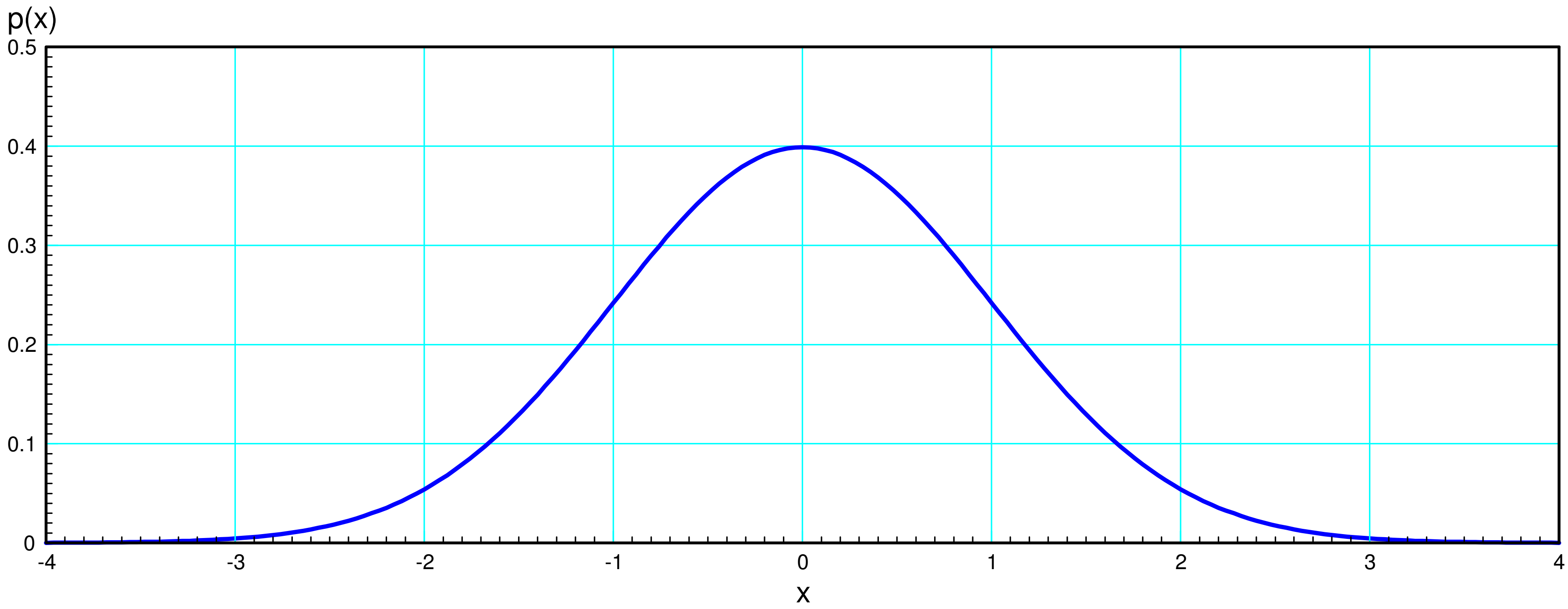
note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

# Normal Distribution

- a.k.a. Gaussian Distribution
- Probably the most important probability distribution
- Class averages, the height or people, IQ scores, etc.




pdf for a normal distribution with a mean of 0, variance of 1

## Central Limit Theorem (coming soon)

- If you sum random variables together (with certain loose restrictions), the resulting distribution converges to a Normal distribution
- If you add to random variables together that have a Normal distribution, the result will have a Normal distribution
- Everything converges to a Normal distribution.
- Once you arrive at a Normal distribution, you're stuck with a Normal distribution.

This is part of the reason Normal distributions are so common.

## pdf and mgf:

A Normal distribution is expressed as

$$X \sim N(\mu, \sigma^2)$$

read as

X has a Normal distribution with mean $\mu$ and variance $\sigma^2$

The pdf for a Normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

while the moment generating function

$$\psi(s) = \exp\left(\mu s + \frac{\sigma^2 s^2}{2}\right)$$

From the moment generating function you can see that

$$m_0 = \psi(0) = 1$$

This is a valid moment generating function

The mean of a Normal distribution is the 1st moment:

$$m_1 = \psi'(0) = \left( (\mu + \sigma^2 s)\exp\left(\mu s + \tfrac{\sigma^2 s^2}{2}\right) \right)_{s=0} = \mu$$

The second moment and variance are

$$m_2 = \psi''(0)$$

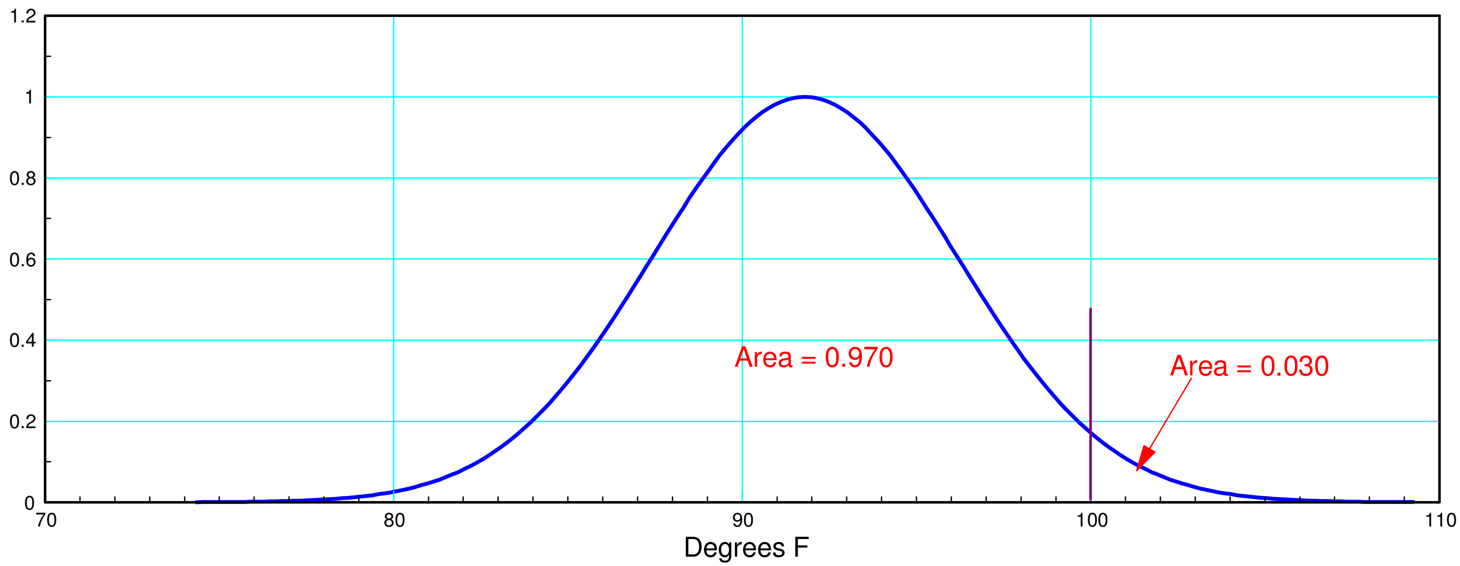
$$= \left( \sigma^2 \exp\left(\mu s + \tfrac{\sigma^2 s^2}{2}\right) + (\mu + \sigma^2 s)^2 \exp\left(\mu s + \tfrac{\sigma^2 s^2}{2}\right) \right)_{s=0} = \sigma^2 + \mu^2$$

$$var = m_2 - m_1^2 = \sigma^2$$

# Examples of Normal Distributions

## Monthly High Temperature:

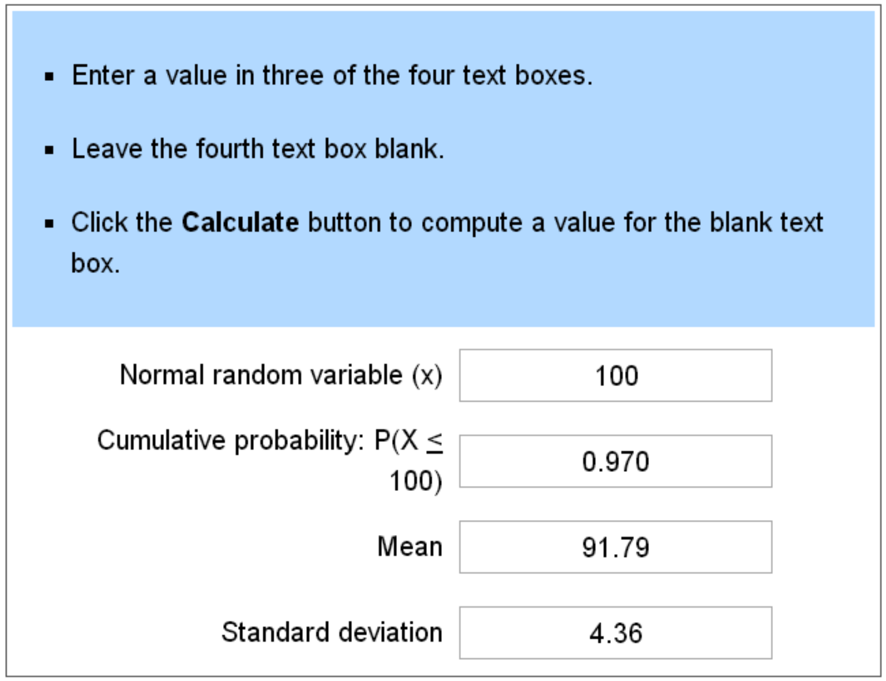
- June, Fargo, ND (source: Hector Airport)
- mean = 91.7962F
- standard deviation = 4.3657




Normalized pdf for the high for the month of June, Fargo ND

## What is the probability that the high will be more than 100F this coming year?

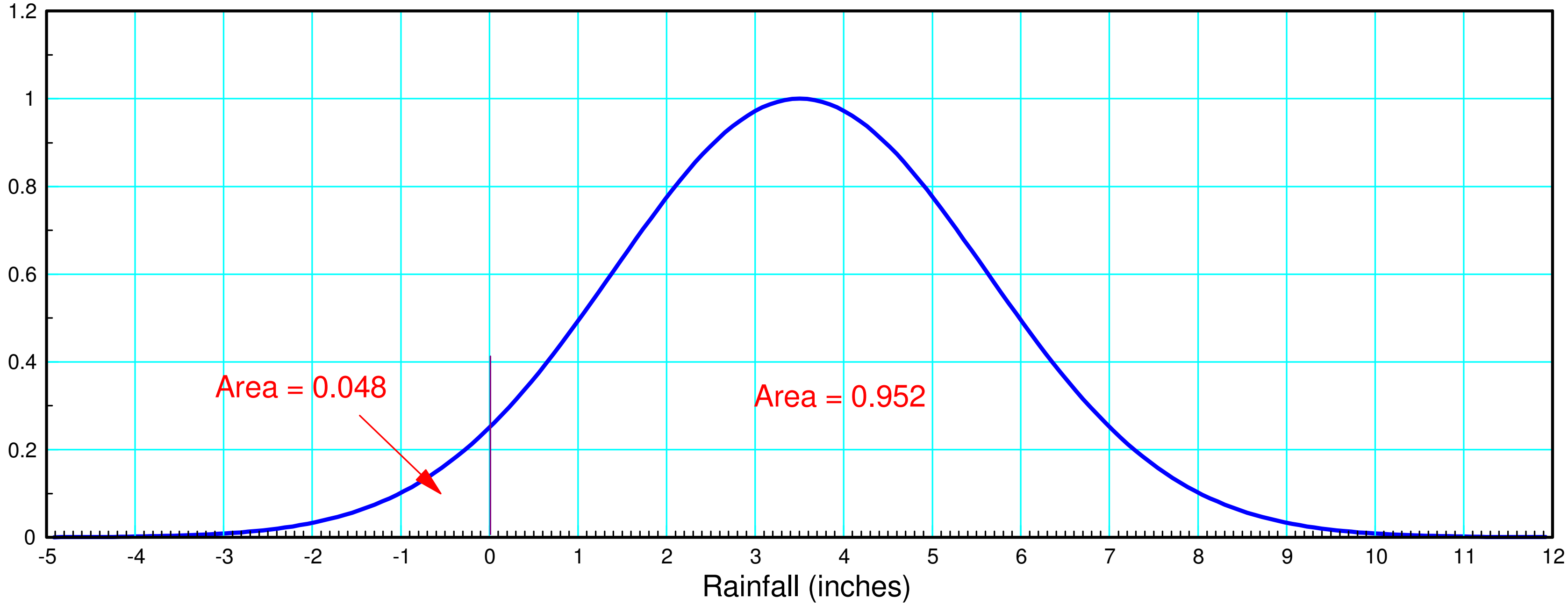
- This is asking what the area of the curve is to the right of 100F.  Here, StatTrek is useful:
- p(high < 100F) = 0.970




https://www.stattrek.com/online-calculator/normal.aspx

**Rain Fall:** (actually a Poisson distribution)

- mean = 3.5025 inches
- standard deviation = 2.1054




pdf for the rainfall in June

## What is the chance we will receive no rain this coming June?

- Sticking with the Normal distribution, the area to the left of 0" is 0.048. There is 4.8% chance we will get no rain this coming June.

- Enter a value in three of the four text boxes.
- Leave the fourth text box blank.
- Click the **Calculate** button to compute a value for the blank text box.

| | |
|---|---|
| Normal random variable (x) | 0 |
| Cumulative probability: P(X ≤ 0) | 0.048 |
| Mean | 3.5025 |
| Standard deviation | 2.105 |

https://www.stattrek.com/online-calculator/normal.aspx

## Addition of Normal Distributions:

If you add to Normal distributions, the result is a Normal distribution with

- mean = mean(a) + mean(b)
- variance = variance(a) + variance(b)

Proof: Multiply the moment generating functions

$$\psi_a(s) = \exp\left(\mu_a s + \frac{\sigma_a^2 s^2}{2}\right) \qquad \psi_b(s) = \exp\left(\mu_b s + \frac{\sigma_b^2 s^2}{2}\right)$$

$$\psi_a(s)\psi_b(s) = \exp\left(\mu_a s + \frac{\sigma_a^2 s^2}{2}\right)\exp\left(\mu_b s + \frac{\sigma_b^2 s^2}{2}\right)$$

$$= \exp\left(\left(\mu_a s + \frac{\sigma_a^2 s^2}{2}\right) + \left(\mu_b s + \frac{\sigma_b^2 s^2}{2}\right)\right)$$

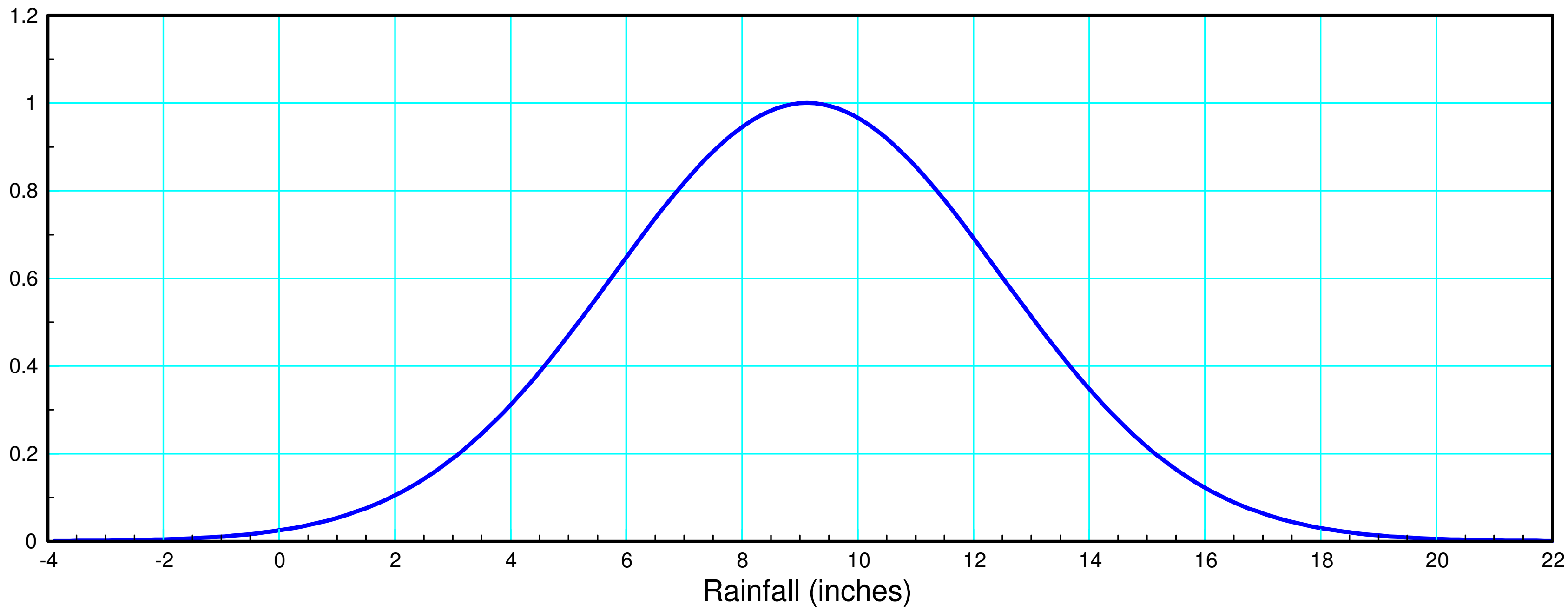
$$= \exp\left((\mu_a + \mu_b)s + \frac{\left(\sigma_a^2 + \sigma_b^2\right) s^2}{2}\right)$$

Example: The rainfall for the months of June, July, and August have the following statistics:

| Month | Mean | Variance |
|---|---|---|
| June | 3.5025 | 4.4327 |
| July | 2.9668 | 3.8044 |
| August | 2.6529 | 3.0063 |
| Sum | 9.1221 | 11.2434 |

What is the mean and variance for the rainfall for the whole summer (June + July + August)

Solution:  Add up the means, add up the variances.




Normalized pdf for the total rainfall over the summer in Fargo

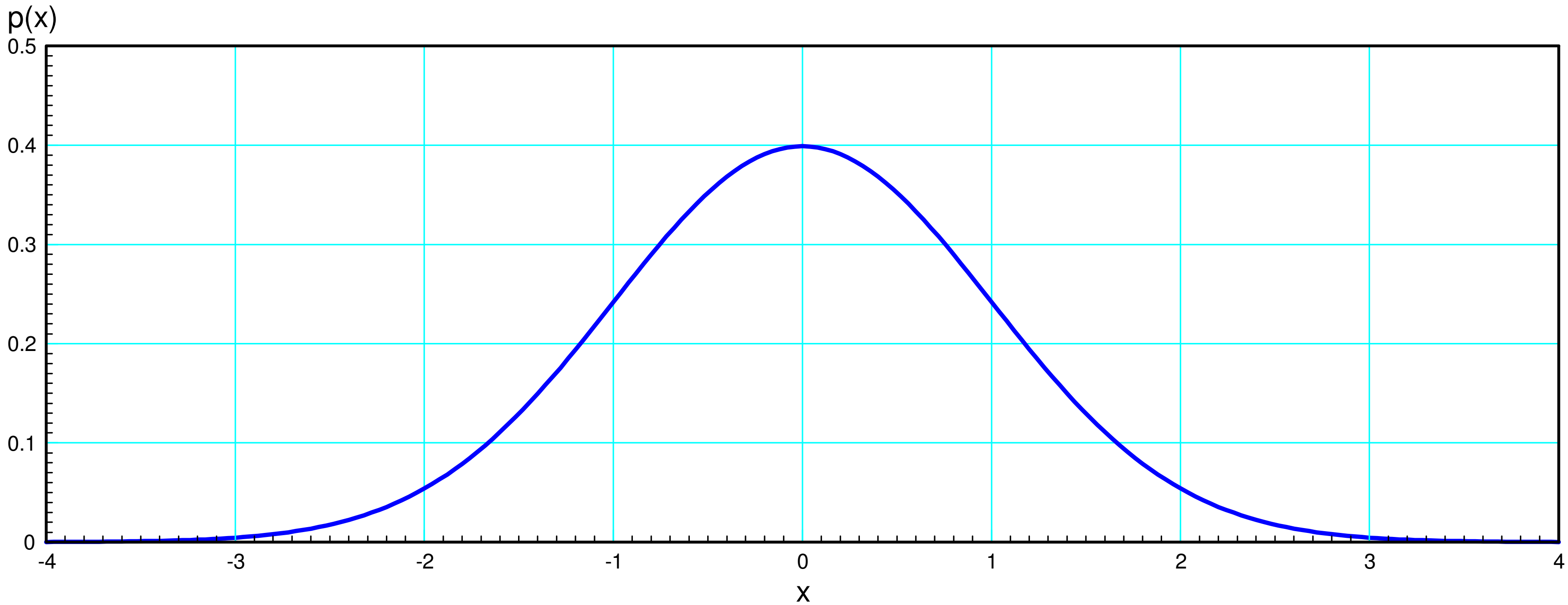
## Standard Normal Distribution

- The Normal distribution is extremely common and extremely useful. Unfortunately, it is difficult to integrate.
- To get around this, tables showing the area under the curve as you move away from the mean could be used. Unfortunately, there are an infinite number of possible means and standard deviations.
- To get around this, the Standard Normal Distribution is used. This is a Normal distribution with
- mean = 0
- standard deviation = 1
- With this stipulation, you can come up with a table showing the area under the curve as you move away from the mean. Once known, you can easily convert this to the distribution you care about.

The standard normal distribution has the following pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2}\right)}$$

It's shape is a nice bell curve:




pdf for a Normal distribution with mean = 0, standard deviation = 1 (a.k.a. a Standard Normal Distribution)

The area under the tail as you move away from the mean is then

| z | 0 | -1 | -2 | -3 | -4 | -5 |
|---|---|---|---|---|---|---|
| p(x<z) | 0.5 | 0.1587 | 0.0227 | 0.001349 | 3.167 10-5 | 2.866 10-7 |

| z | -1.28 | -1.64 | -1.96 | -2.24 | -2.33 | -2.58 |
|---|---|---|---|---|---|---|
| p(x<z) | 0.1 | 0.05 | 0.025 | 0.0125 | 0.01 | 0.005 |

The z-score is how far your value is from the mean in terms of standard deviations

$$z = \left(\frac{x-\mu}{\sigma}\right)$$

This converts your data (x) into a standard Normal distribution (z).

Example: Recall that the high for the month of June has

- mean = 91.7962F
- standard deviation = 4.3657

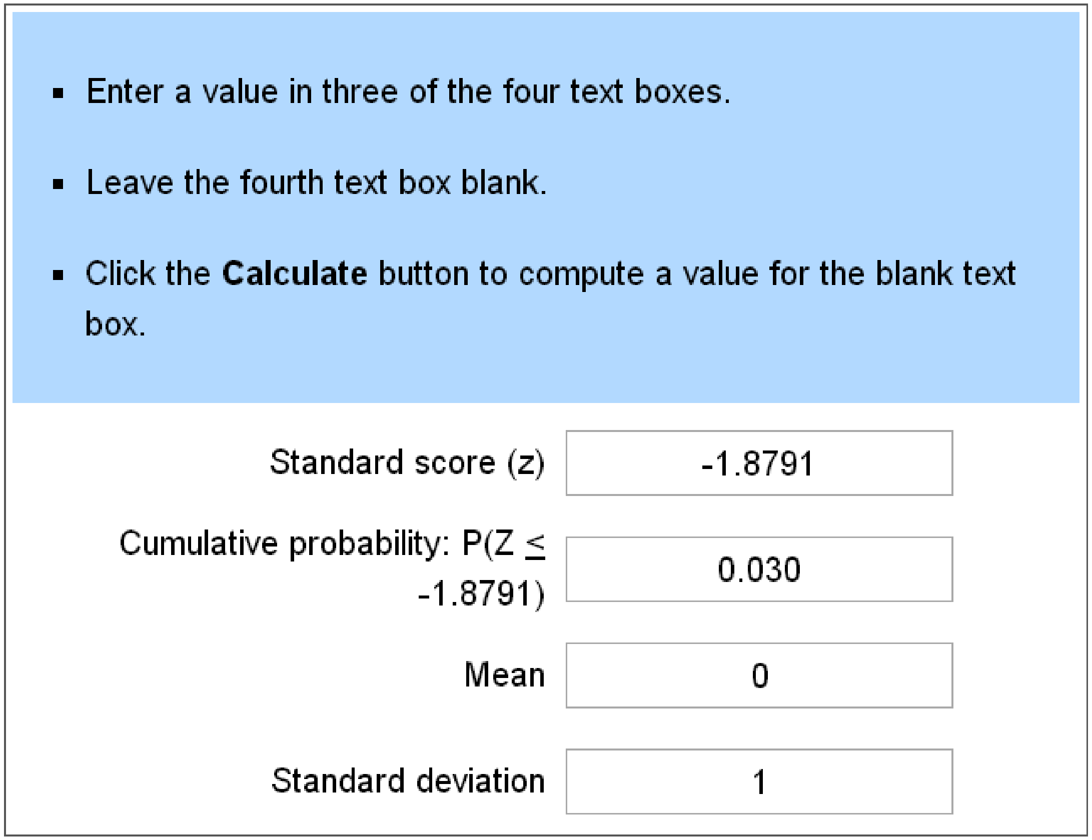
Determine the probability that it will break 100F this coming June.

Solution: Determine the z-score corresponding to 100F

$$z = \left(\frac{100F - 91.7962}{4.3658}\right) = 1.8791$$

From the table at the end, 1.8751 has an area ( p(x<z) ) of 0.0301

You can also use StatTrek



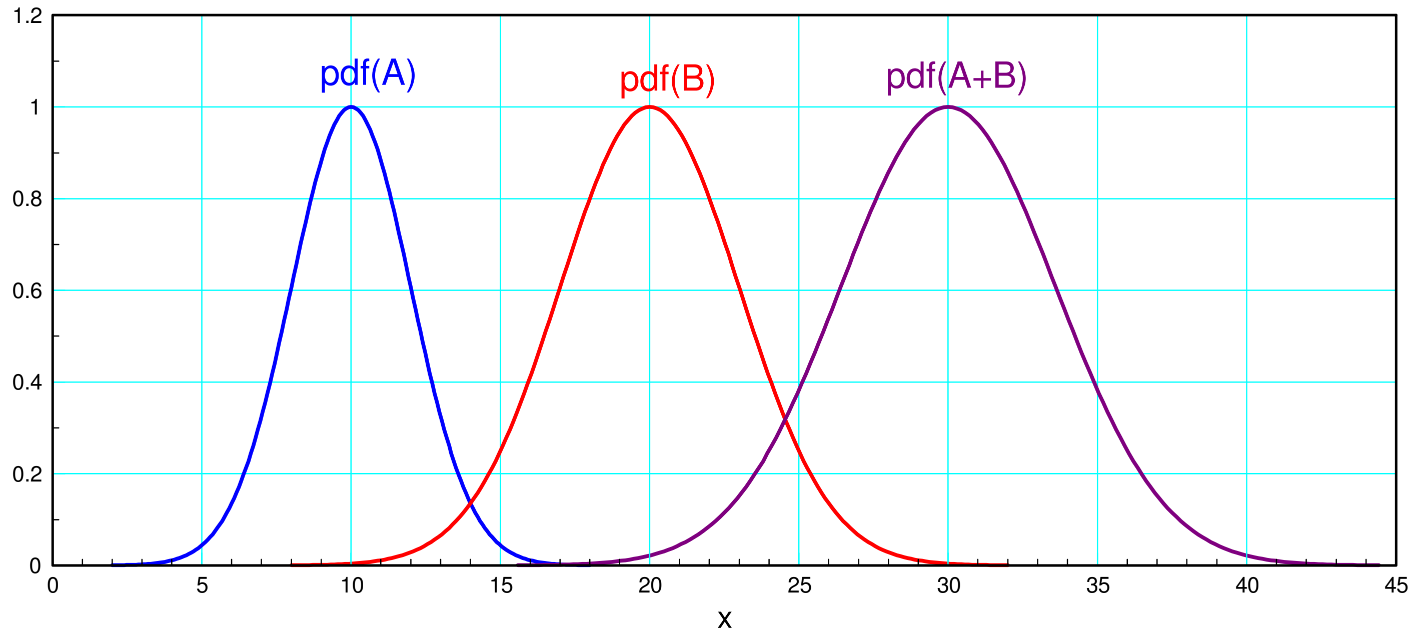

https://www.stattrek.com/online-calculator/normal.aspx

The area is 0.030 or 3.0% (same answer as before)

# Adding of Normal Distributions:

When you add normal distributions,

- The result is a normal distribution,
- The means add, and
- The variance adds

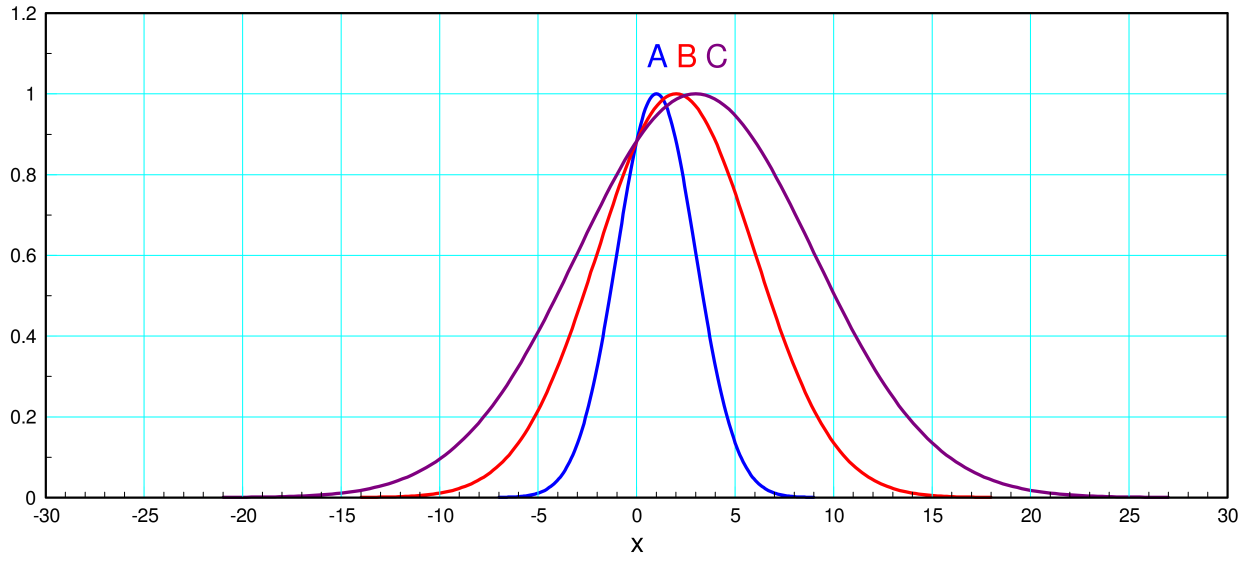


## Problem: Let

- a be a normal distribution with a mean of 1 and standard deviation of 2
- b be a normal distribution with a mean of 2 and standard deviation of 4
- c be a normal distribution with a mean of 3 and standard deviation of 6

Find

- The distribution of $y = a + b + c$
- The probability that $y > 10$



## Distribution of y

y will have a normal distribution.

Mean:

$$\mu_y = \mu_a + \mu_b + \mu_c$$

$$\mu_y = 1 + 2 + 3 = 6$$

Variance:

$$\sigma_y^2 = \sigma_a^2 + \sigma_b^2 + \sigma_c^2$$

$$\sigma_y^2 = 2^2 + 4^2 + 6^2 = 56$$

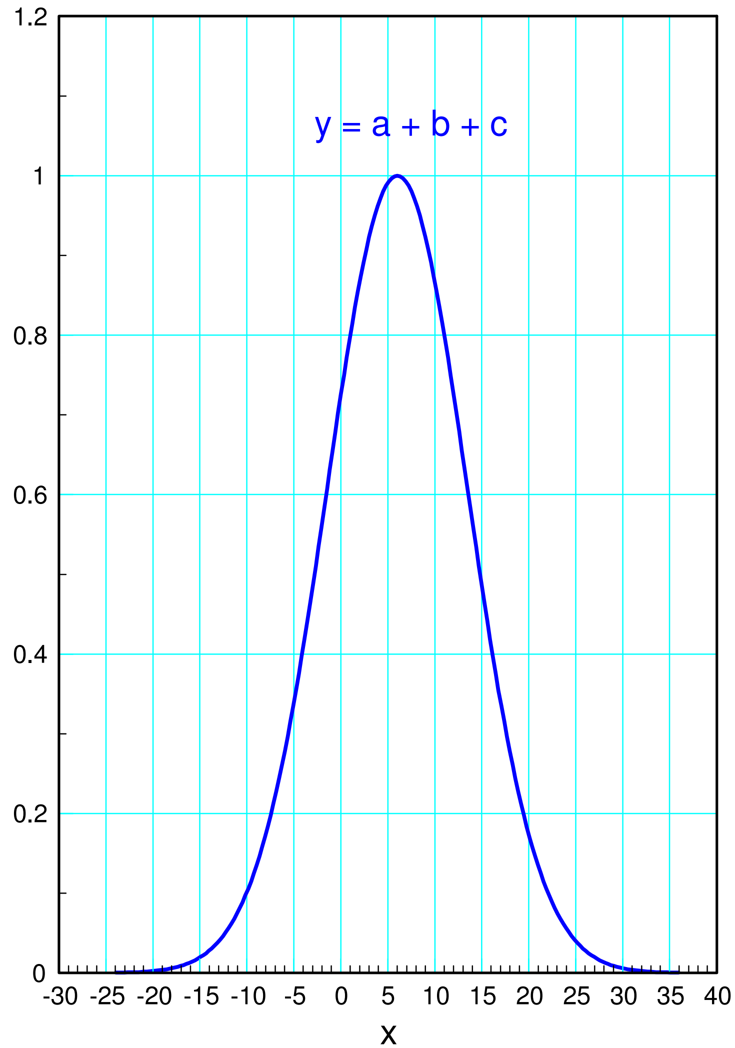
Standard Deviation:

$$\sigma_y = \sqrt{56}$$

# Checking with a Monte-Carlo simulation

```
n = 1e5;
y = zeros(n,1);
for i=1:n
    a = 2*randn + 1;
    b = 4*randn + 2;
    c = 6*randn + 3;
    y(i) = a + b + c;
end
disp('    mean    variance   std')
disp([mean(y), var(y), std(y)])
```

```
    mean     variance    std
    6.0141    56.0709     7.4881
```




Check:

- The mean adds
- The variance adds

# Probability y > 10:

Find the z-score

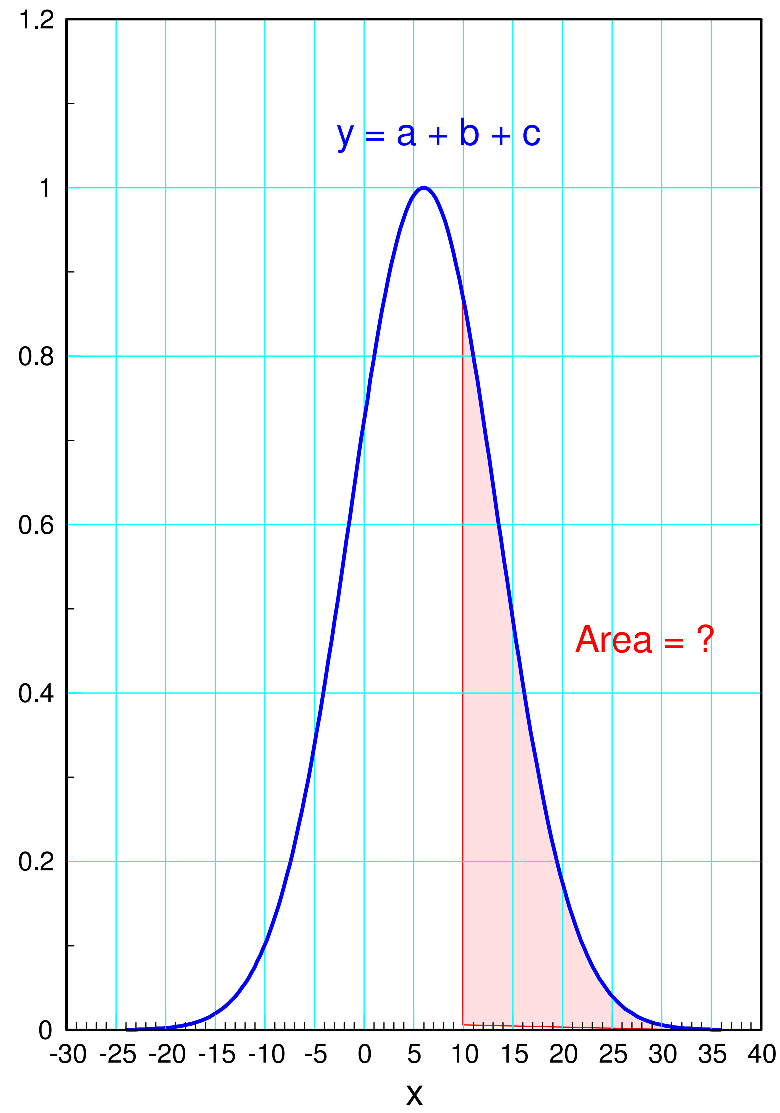
$$z = \left(\frac{10 - \mu_y}{\sigma_y}\right)$$

$$z = \left(\frac{10 - 6}{\sqrt{56}}\right) = 0.5345$$

Convert to a probability using a standard normal table

- tail area = 0.2965

There is a 29.65% chance that the sum will be more than 10.0



# Checking with a Monte-Carlo simulation

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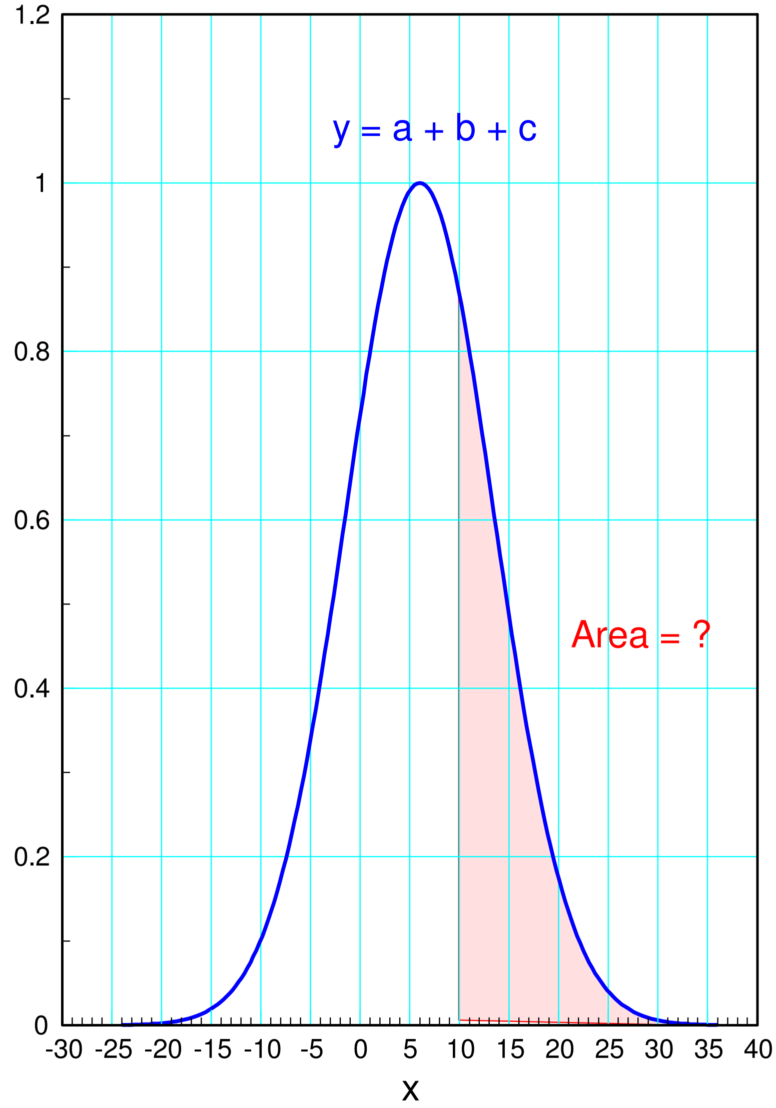
    c = 6*randn + 3;
    y(i) = a + b + c;
end
disp('    mean     variance   std')
disp([mean(y), var(y), std(y)])
```

```
    mean     variance    std
    6.0141    56.0709     7.4881
```

```
sum(y > 10)

ans =     29818
```




## $p(y > 10) = 29.818\%$

- 29.65% chance calculated

# Summary

Normal distributions are fairly easy to work with

- Normal + Normal = Normal
- The means add
- The variance adds

Probabilies are defined by the distance to the mean

- z-score

With a normal table, you can determine

- The area of a tail (one-sided test), or
- The area of two tails (two-sided test)

# Standard Normal Table: z-score

| | 0 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **-2** | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| **-1.9** | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.025 | 0.0244 | 0.0239 | 0.0233 |
| **-1.8** | 0.0359 | 0.0352 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| **-1.7** | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| **-1.6** | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| **-1.5** | 0.0668 | 0.0655 | 0.0643 | 0.063 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| **-1.4** | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0722 | 0.0708 | 0.0694 | 0.0681 |
| **-1.3** | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| **-1.2** | 0.1151 | 0.1131 | 0.1112 | 0.1094 | 0.1075 | 0.1057 | 0.1038 | 0.102 | 0.1003 | 0.0985 |
| **-1.1** | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.123 | 0.121 | 0.119 | 0.117 |
| **-1** | 0.1587 | 0.1563 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| **-0.9** | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.166 | 0.1635 | 0.1611 |
| **-0.8** | 0.2119 | 0.209 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| **-0.7** | 0.242 | 0.2389 | 0.2358 | 0.2327 | 0.2297 | 0.2266 | 0.2236 | 0.2207 | 0.2177 | 0.2148 |
| **-0.6** | 0.2743 | 0.2709 | 0.2676 | 0.2644 | 0.2611 | 0.2579 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| **-0.5** | 0.3085 | 0.305 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.281 | 0.2776 |
| **-0.4** | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.33 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| **-0.3** | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.352 | 0.3483 |
| **-0.2** | 0.4207 | 0.4168 | 0.4129 | 0.4091 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| **-0.1** | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| **0** | 0.5 | 0.496 | 0.492 | 0.488 | 0.4841 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |