Gamma and Poisson Distribution

ECE 341: Random Processes

Lecture #16

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Gamma and Poisson Distribution

Both are extensions of the exponential distribution:

$$p(t) = ae^{-at}$$

Gamma: Time until k events occur.

- The time until k customers arrive,
- The time until k atoms decay,
- The time until you've been invited to k parties,

Poisson: Probability k events occur in M seconds

- The number of pieces of mail you receive each day (the sending time is exponential)
- The number of cars through in intersection in one minute
- The number of customers arriving at a restaurant in one hour
- Also used to approximat binomial distributions where n is large

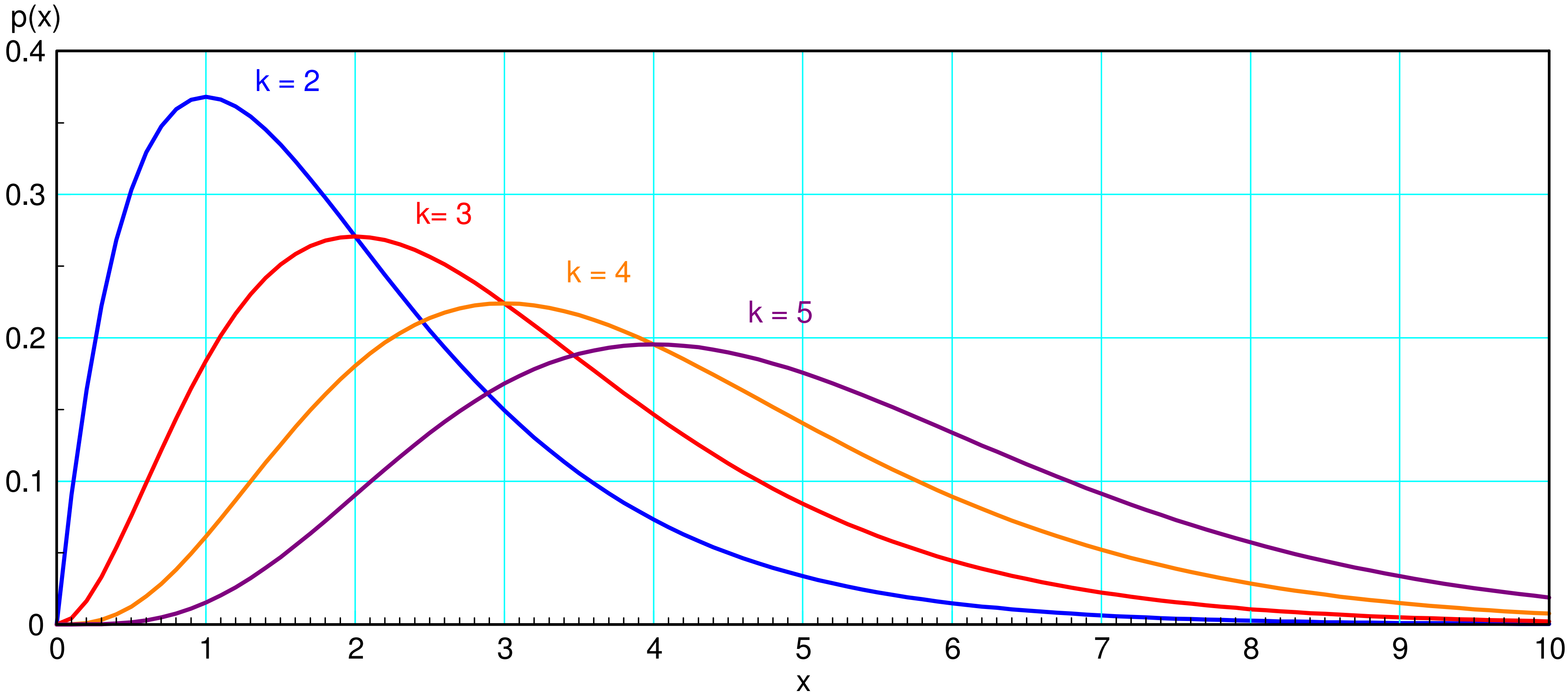
Gamma Distribution

Exponential: Time until next event

Gamma: Time until k events

	Exponential	Gamma
pdf	$f_x = a\ e^{-ax}$	$f_x = \left(\frac{a^k}{(k-1)!}\right)\ x^{k-1}\ e^{-ax}$
cdf	$F_x = 1 - e^{-ax}$	complicated
mgf	$\left(\frac{a}{s+a}\right)$	$\left(\frac{a}{s+a}\right)^k$
mean	$\left(\frac{1}{a}\right)$	$\left(\frac{k}{a}\right)$
variance	$\left(\frac{1}{a^2}\right)$	$\left(\frac{k}{a^2}\right)$

pdf of a Gamma distribution:



pdf for a Gamma distribution with an average arrival time of 1

Derivation of pdf from mgf:

Example 1: Determine the pdf and cdf for a Gamma distribution with

- k = 3, and
- a = 0.2

Solution: The moment generating functionis

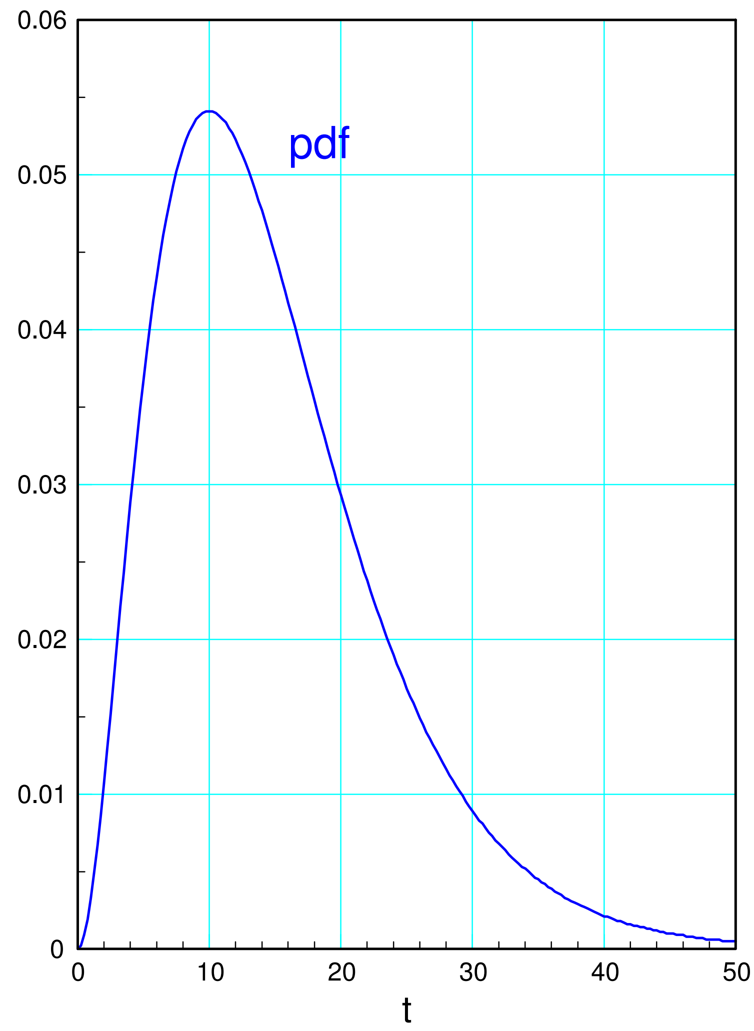
$$\psi(s) = \left(\frac{0.2}{s+0.2}\right)^3$$

From a table of LaPlace transforms

$$\left(\frac{1}{s-a}\right)^3 \rightarrow \frac{1}{2!}\, t^2\, e^{at}\, u(t)$$

Substituting

$$f_x = (0.2)^3 \cdot 2\ \cdot \frac{1}{2}\, t^2\, e^{-0.2t}\, u(t)$$



The cdf is the integral of the pdf

$$F_X(s) = \left(\frac{0.2}{s+0.2}\right)^3 \left(\frac{1}{s}\right)$$

Doing partial fraction expansion

$$F_X(s) = \left(\frac{a}{s}\right) + \left(\frac{b}{(s+0.2)^3}\right) + \left(\frac{c}{(s+0.2)^2}\right) + \left(\frac{d}{s+0.2}\right)$$

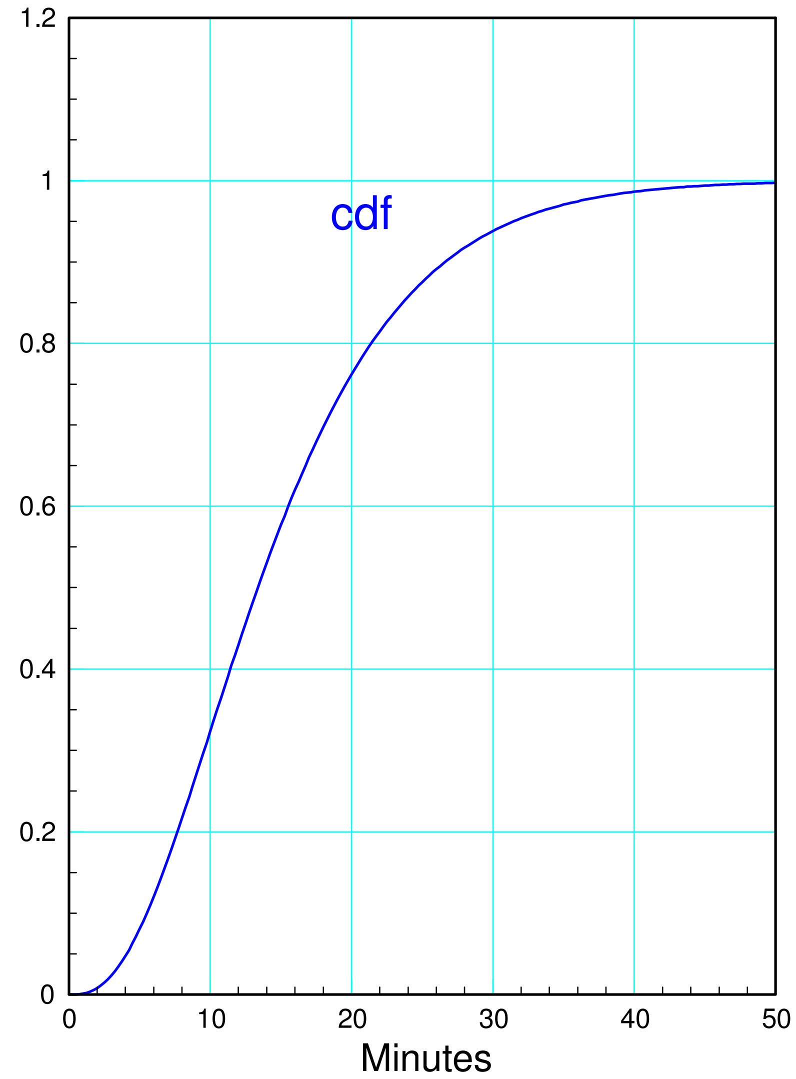
$$F_X(s) = \left(\frac{1}{s}\right) + \left(\frac{-0.04}{(s+0.2)^3}\right) + \left(\frac{-0.2}{(s+0.2)^2}\right) + \left(\frac{-1}{s+0.2}\right)$$

Terms

- {a, b} are from the cover-up method
- {c,d} are from placing over a common denominator

The cdf is thus

$$F_x = (1 + (-0.02\ t^2 - 0.2\ t - 1)\ e^{-0.2t})\ u(t)$$



Gamma Distribution with Learning

Assume you are doing a task

- Each attempt has an exponential distribution

You get better each time (learning is going on)

- 1st attempt: mean = 10 minutes
- 2nd attemtp: mean = 9 minutes
- 3rd attempt: mean = 8 minutes

How long will it take you to complete three tasks?

What is the probability you will complete all three tasks within 30 minutes?

Moment Generating Functions

This is where moment generating functions really shine:

The LaPlace transform for each task is

$$A(s) = \left(\frac{1/10}{s+1/10}\right) = \left(\frac{1}{10s+1}\right)$$

$$B(s) = \left(\frac{1/9}{s+1/9}\right) = \left(\frac{1}{9s+1}\right)$$

$$C(s) = \left(\frac{1/8}{s+1/8}\right) = \left(\frac{1}{8s+1}\right)$$

The LaPlace transform for completing all three tasks is

$$Y = ABC = \left(\frac{1/10}{s+1/10}\right)\left(\frac{1/9}{s+1/9}\right)\left(\frac{1/8}{s+1/8}\right)$$

pdf of 3 tasks

The pdf is the inverse tranform.

$$Y(s) = \left(\frac{1/10}{s+1/10}\right)\left(\frac{1/9}{s+1/9}\right)\left(\frac{1/8}{s+1/8}\right)$$

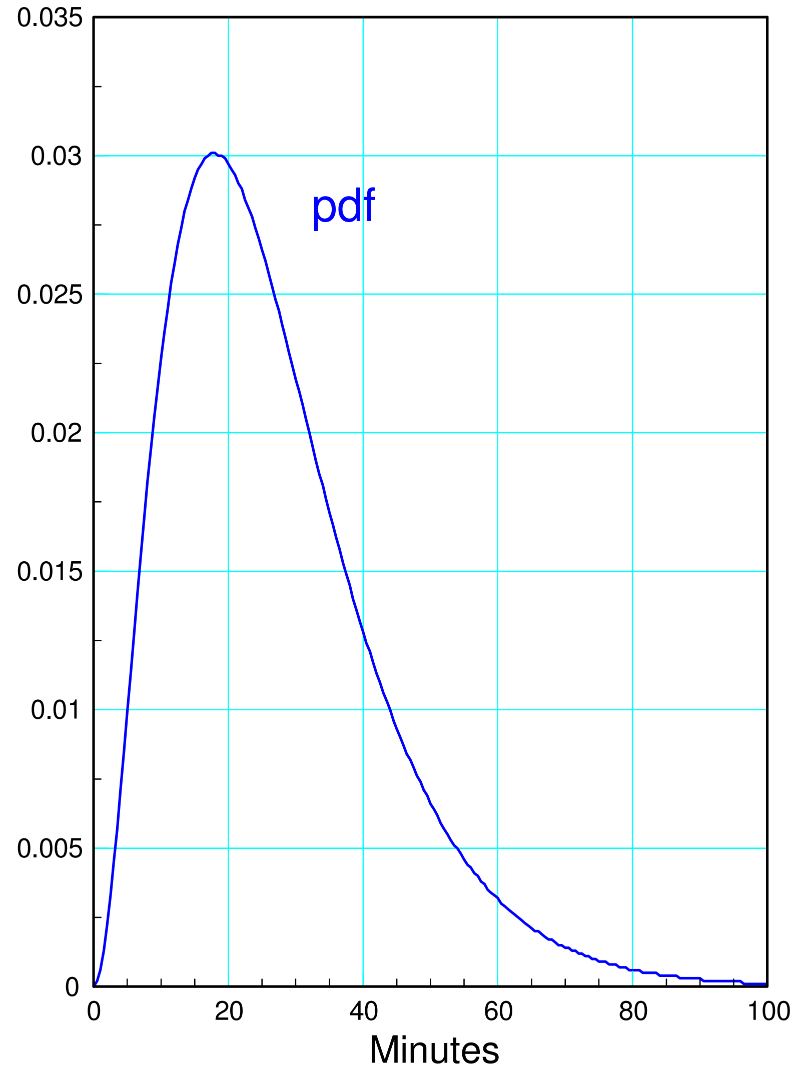
Take a partial fraction expansion

$$Y(s) = \left(\frac{5}{s+1/10}\right) + \left(\frac{-9}{s+1/9}\right) + \left(\frac{4}{s+1/8}\right)$$

Take the inverse LaPla

ce transform

$$y(t) = (5e^{-t/10} - 9e^{-t/9} + 4e^{-t/8})u(t)$$



cdf of Three Tasks

The cdf is the integral of the pdf

In LaPlace

$$cdf = \left(\frac{1}{s}\right) pdf$$

or

$$cdf = \left(\frac{1}{s}\right)\left(\frac{1/10}{s+1/10}\right)\left(\frac{1/9}{s+1/9}\right)\left(\frac{1/8}{s+1/8}\right)$$

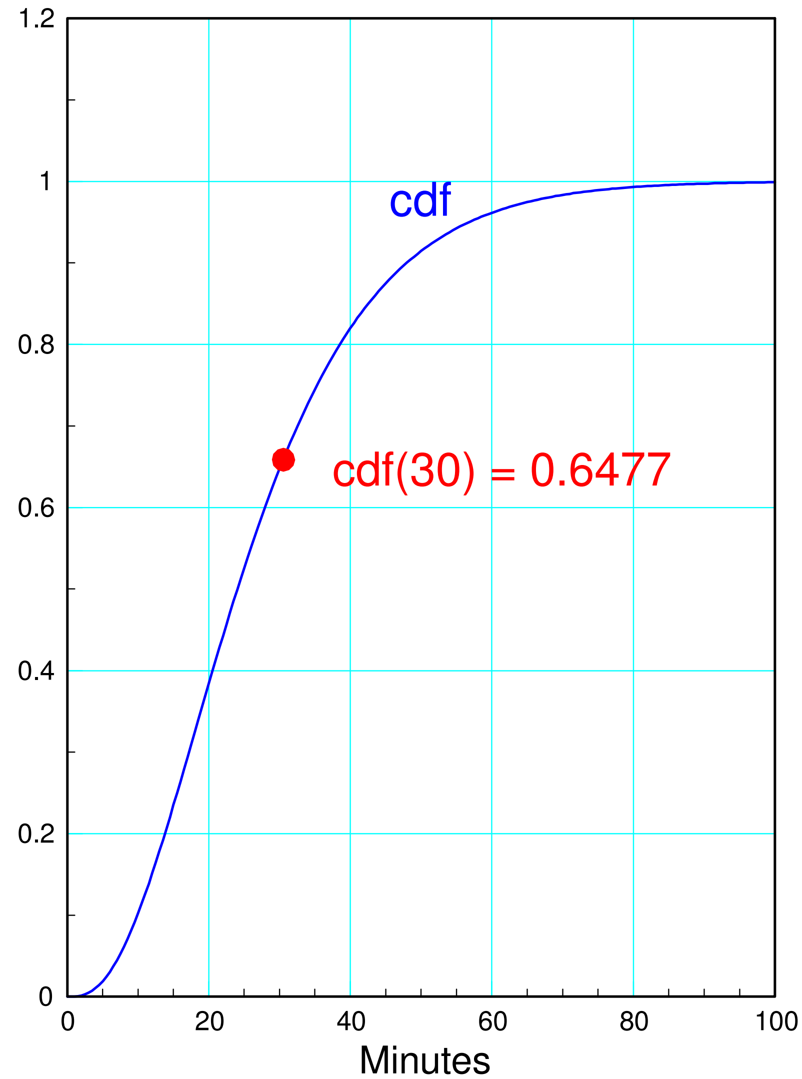
Do a partial fraction expansion

$$cdf = \left(\frac{1}{s}\right) + \left(\frac{-50}{s+1/10}\right) + \left(\frac{81}{s+1/9}\right) + \left(\frac{-32}{s+1/8}\right)$$

gives

$$cdf = (1 - 50e^{-t/10} + 81e^{-t/9} - 32e^{-t/8})u(t)$$

$$cdf(30) = 0.6477$$



Poisson Distribution.

The Poisson distribution is actually a discrete probability function.

- Gamma: The time until the kth customer arrives, (Gamma)
- Poisson: The probability that k customers will arrive in a fixed interval

The Poisson distribution is useful if you want to know

- How many cars will go through an intersection in one hour,
- How many customers will arrive in one hour,
- How many patients will go to the emergency room in one day, or
- The number of times your boss will notice you over the course of one week.
- The probability of a binomial distribution when n is large

Assumptions

(Wikipedia)

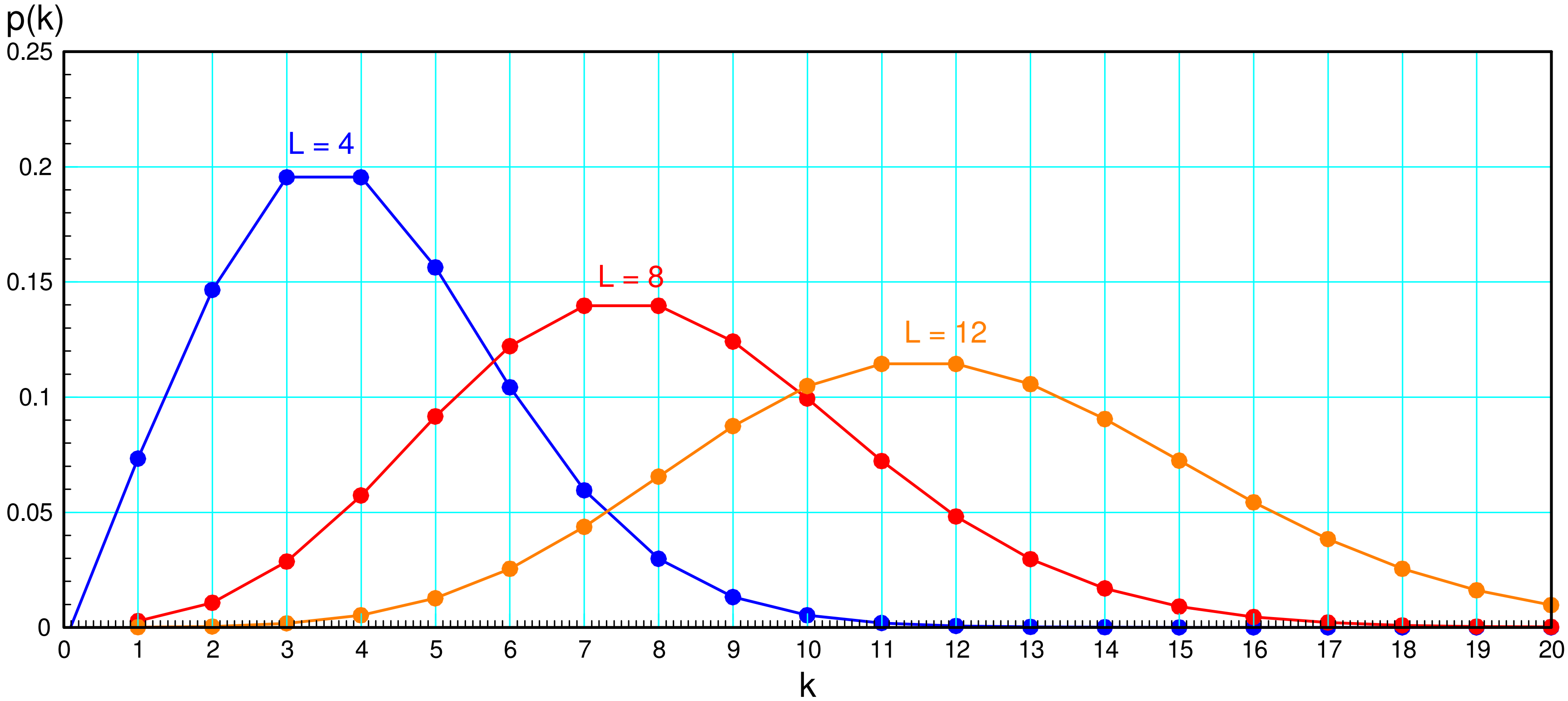
- k is the number of times an event occurs in an interval and k can take values 0, 1, 2, ...
- The occurrence of one event does not affect the probability that a second event will occur. That is, events occur independently.
- The average rate at which events occur is independent of any occurrences. For simplicity, this is usually assumed to be constant, but may in practice vary with time.
- Two events cannot occur at exactly the same instant; instead, at each very small subinterval exactly one event either occurs or does not occur.

The pdf for a Poisson distribution is (wikipedia)

	Exponential	Poisson
pdf	$f_x = a\ e^{-ax}$	$f(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$
cdf	$F_x = 1 - e^{-ax}$	complicated
mgf	$\left(\frac{a}{s+a}\right)$	$\exp\left(\lambda(z-1)\right)$
mean	$\left(\frac{1}{a}\right)$	λ
variance	$\left(\frac{1}{a^2}\right)$	λ

Here, $\lambda = 1/a$ for equating exponential and Poisson processes.

The pdf for a Poisson distribution looks like the following ($\lambda = 10$ (a = 1/10) for illustration purposes).



pdf for a Poisson distribution with λ = {4, 8, 12}.
Note that this is a discrete pdf - so only the integer values of k matter

Poisson Approximation for a Binomial Distribution

A Poisson distribution is also a good approximation for a binomial distribution where

- n is large
- $\lambda = np$

Example 1: The pdf for a binomial distribution

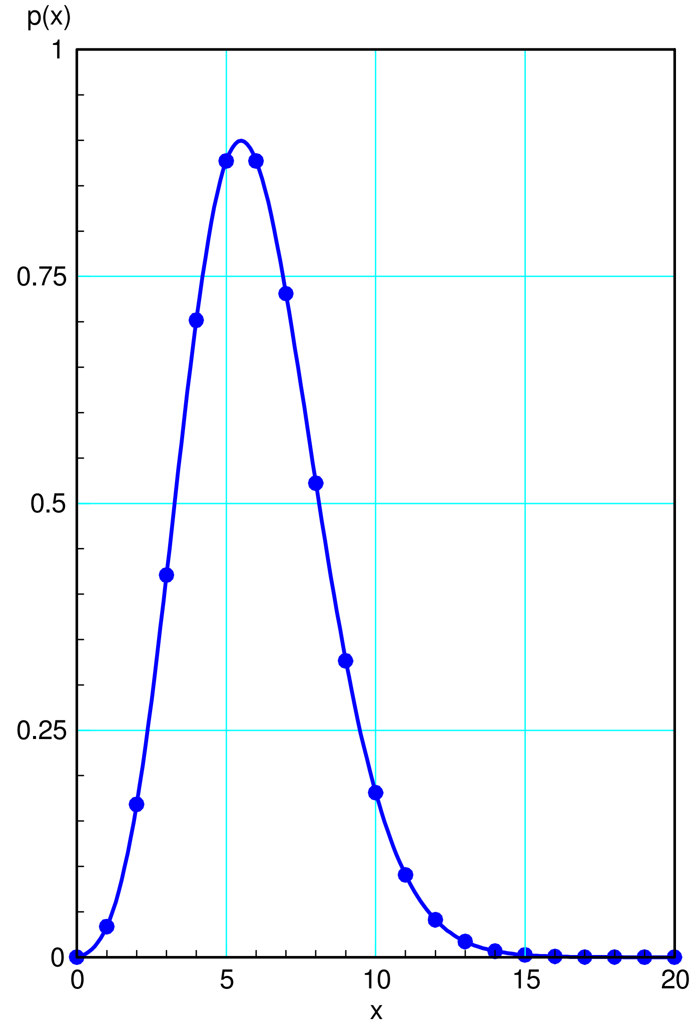
$$n = 100, \qquad p = 0.05, \qquad \lambda = np = 5$$

Binomial:

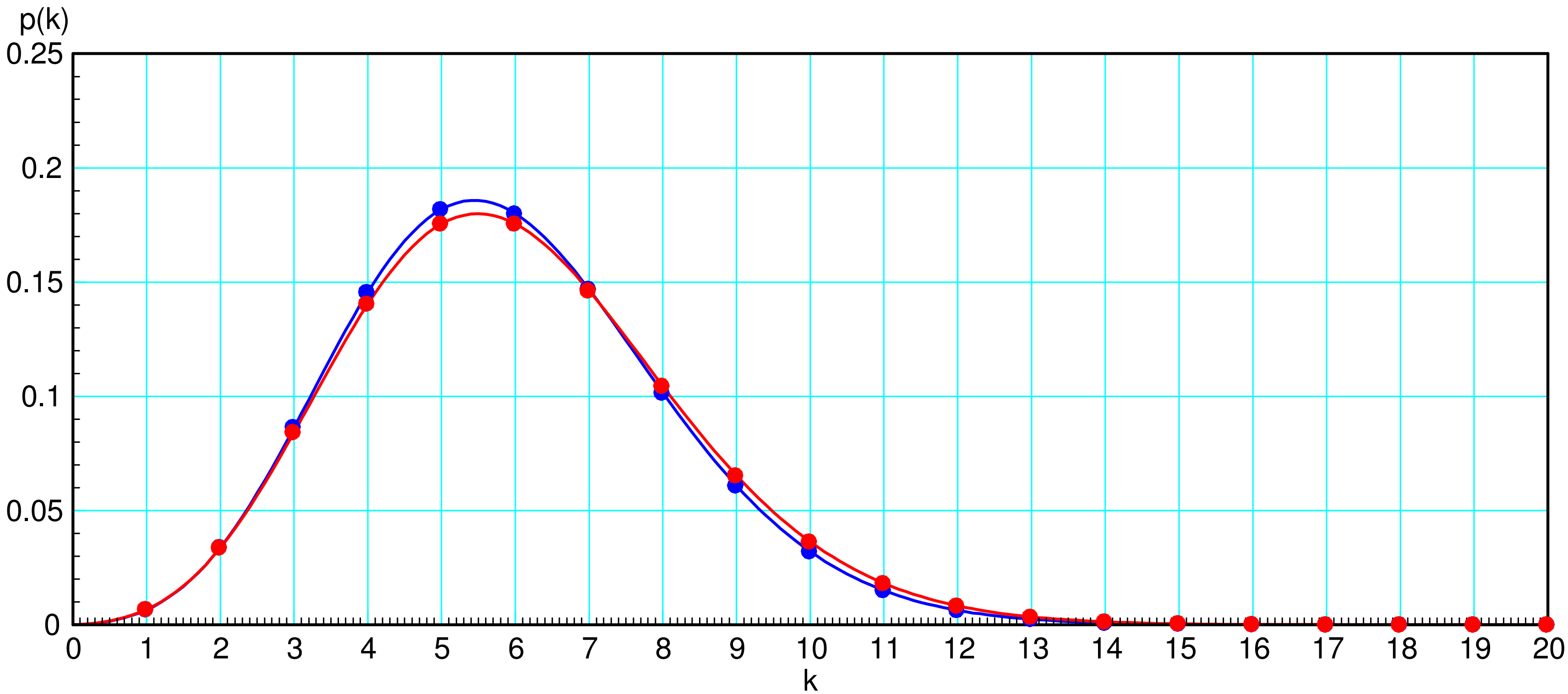
$$f_1(x) = \binom{100}{x}(0.05)^x(0.95)^{100-x}$$

Poisson ($\lambda = np = 5$)

$$f_2(x) = \frac{1}{x!} \cdot 5^x \cdot e^{-5}$$



Poisson vs. Binomial



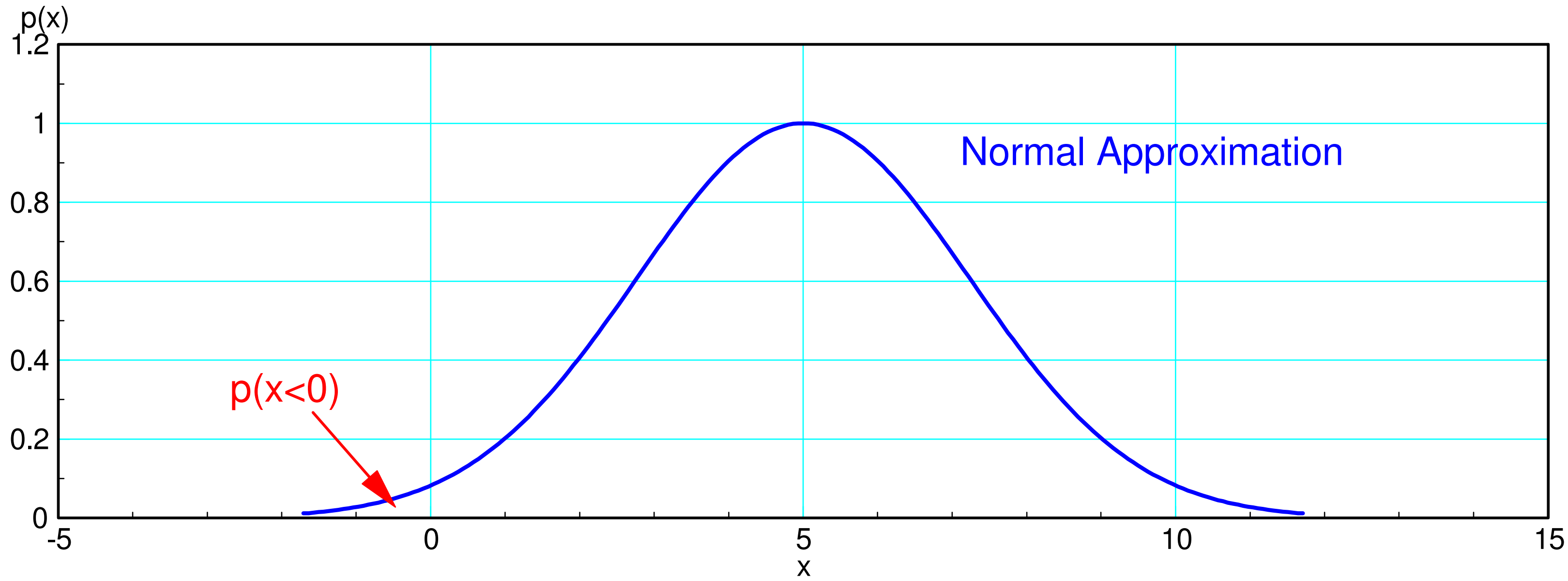
Binomial (blue) vs. Poisson (red) with np = 5

A Poisson approximation is a slightly more complicated approximation for a binomial distribution than a Normal approximation. It's more accurate however.

- A normal distribution goes from -∞ to +∞
- A Poisson distribution is zero for k < 0

In the case of a binomial distribution, you'll never get a negative total.

- A Poisson approximation is slightly more accurate.



Example 2: Plot the probability density function for a binomial distribution with

n = 10,000   p = 0.000 5   np = 5

A Binomial pdf has numeric problems:

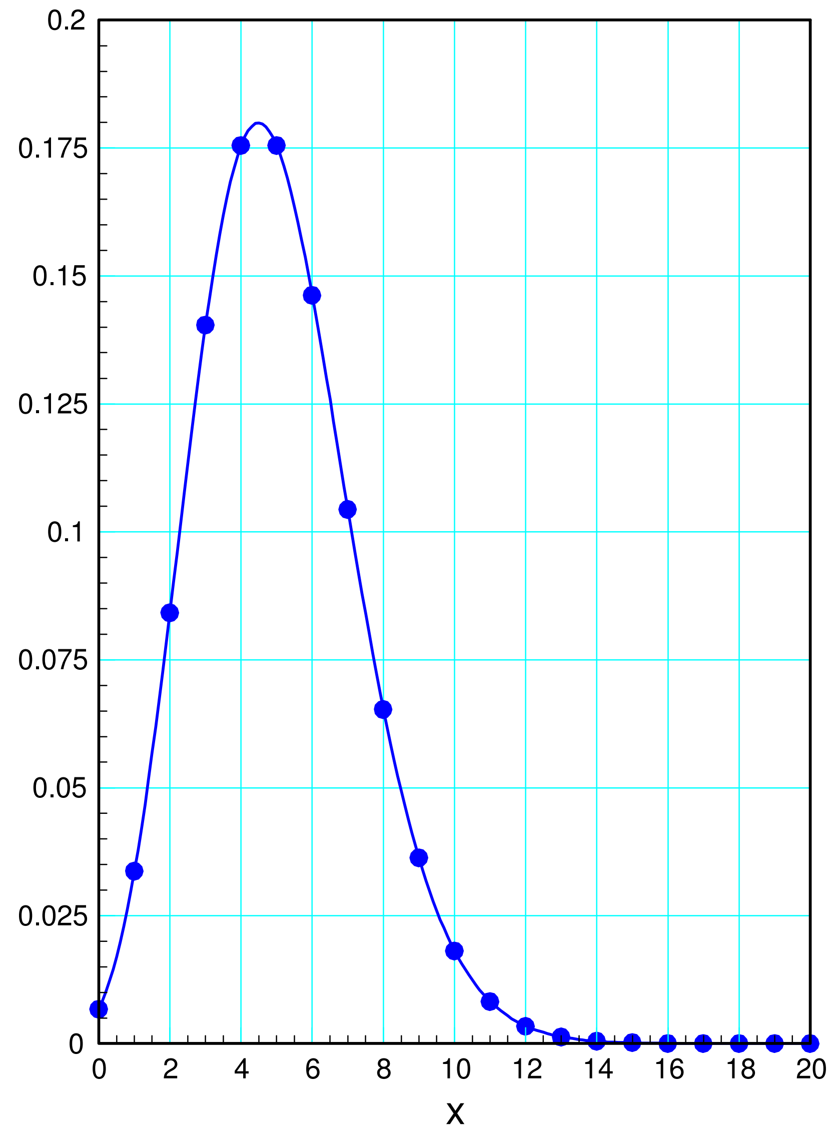
- 10,000! is out of range

$$f(x) = \binom{10,000}{x} (0.0005)^x (0.9995)^{10,000-x}$$

Not a problem with a Poisson approximation

- np = 5

$$f(x) \approx \frac{1}{x!} \cdot 5^x \cdot e^{-5}$$



Summary

A Gamma distribution is an exponential distribution

- Where you wait until N events occur
- The moment generating function is $\left(\frac{a}{s+a}\right)^N$

A Poisson distribution

- Is an exponential distribution where you count how many events occur over a time interval
- Is a good approximation for a binomial distribution