Student t-Test

ECE 341 Random Processes

Jake Glower - Lecture #25

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Calculating Probabilities

In this course, we've covered several ways to calculate probabilities.

For example, let

$$Y = 4d4 + 3d6 + 2d8$$

Determine

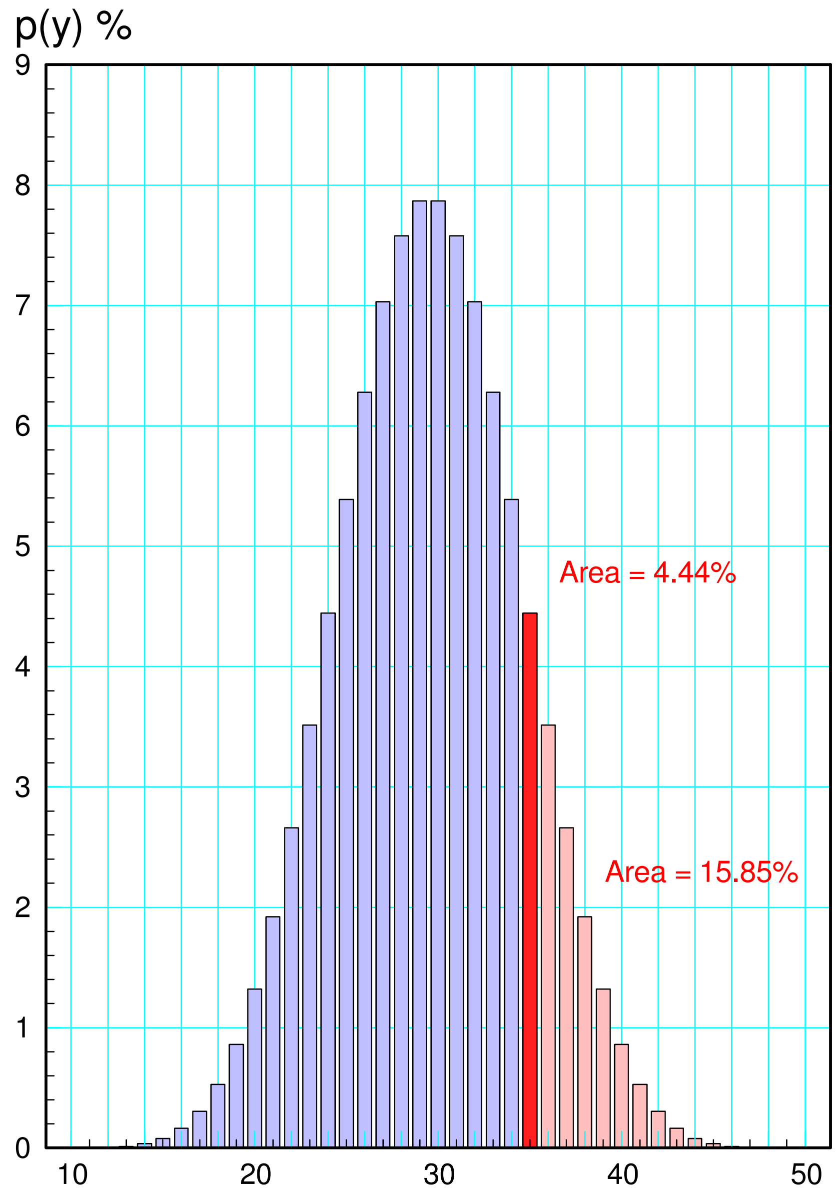
- $p(y = 35)$
- $p(y > 35)$
- 90% confidence interval for y

Option 1: Monte-Carlo

Roll the dice 1 million times

Count the results

- pdf shown to the right



Results:

- p(y = 35) = 4.44%
- p(y > 35) = 15.85%
- p(21.5 < y < 36.5) = 90%

note:

- This took 1,000,000 rolls
- At $1/roll, this costs $1,000,000

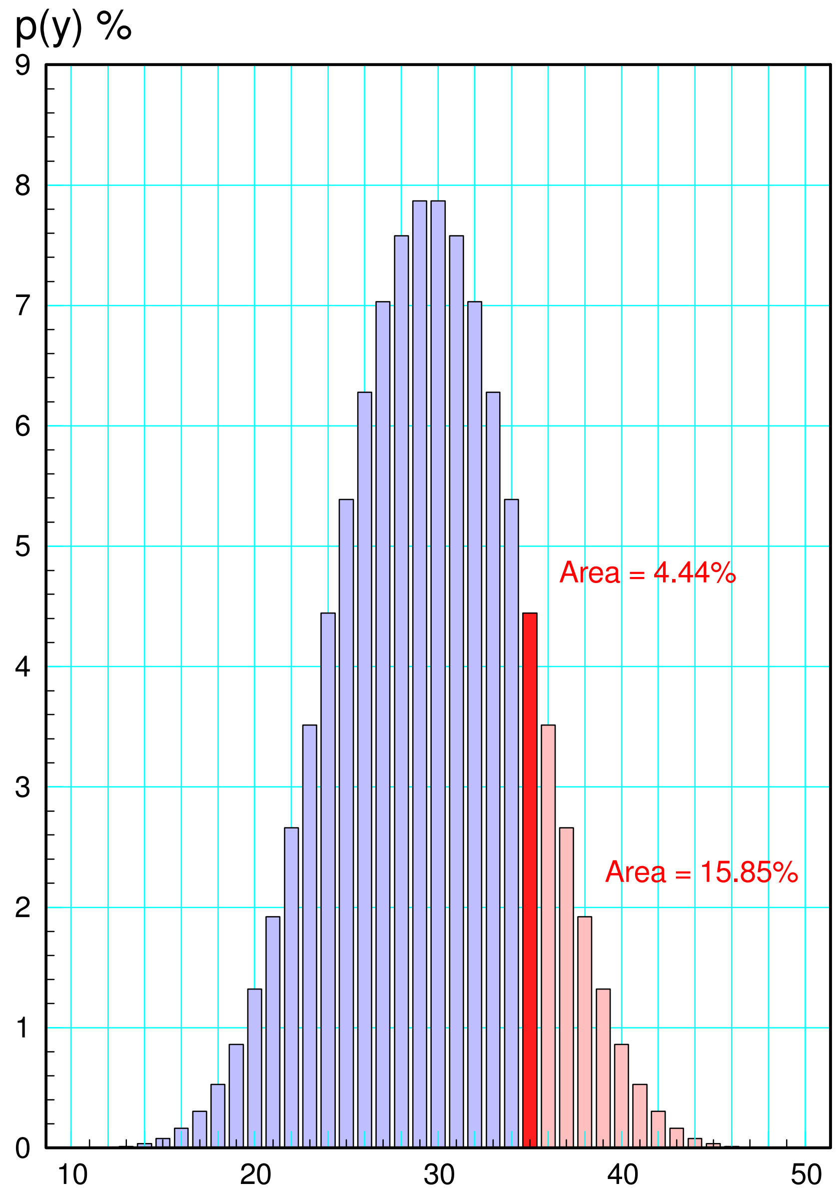
Option 2: Calculate the odds

Several ways to do this

- Enumeration
- Convolution
- Combinatorics

These all give exact answers

You have to know the pdf for these to work, however



Option 3: Normal Approximation

We *know* the mean and variance for a single die

We can *calculate* the mean and variance for y

	d4	d6	d8	4d4 + 3d6 + 2d8
mean	2.5	3.5	4.5	29.5
variance	1.25	2.9167	5.25	24.25

Normal Approximation (cont'd)

Once you know the mean and variance, you can calculate the odds

- Determine the z-score
- From a t-Table convert to a probability

$p(y < 34.5) = 0.8450$

$p(y < 35.5) = 0.9995$

so

$p(34.5 < y < 35.5) = 0.0434$

90% confidence interval

- $21.4 < y < 37.6$

```
>> z1 = (34.5-29.5)/sqrt(24.25)
z1 =    1.0153

>> p1 = (erf(z1/sqrt(2)) + 1)/2
p1 =    0.8450

>> z2 = (35.5-29.5)/sqrt(24.25)
z2 =    1.2184

>> p2 = (erf(z2/sqrt(2)) + 1)/2
p2 =    0.8885

>> p = p2 - p1
p =    0.0434

>> p3 = (erf(z3/sqrt(2)) + 1)/2
p3 =    0.1115

>> 29.5 + 1.64485 * sqrt(24.25)
ans =   37.5999

>> 29.5 - 1.64485 * sqrt(24.25)
ans =   21.4001
```

Normal Approximation: Results

With a Normal approximation, you get

- Almost the same results
- With zero die rolls

	Monte-Carlo	Normal Approx
p(y = 35)	4.43%	4.34%
p(y > 35)	11.39%	11.15%
90% confidnce interval	[22, 37]	(21.4, 37.6)
# rolls	1,000,000	0

Problem

What if

- You don't know the pdf?
- You don't know the mean?
- You don't know the variance?

But you can collect measurements...

Solution (Option 4): Student t Distribution

Assume a Normal distibution

- Usually the case

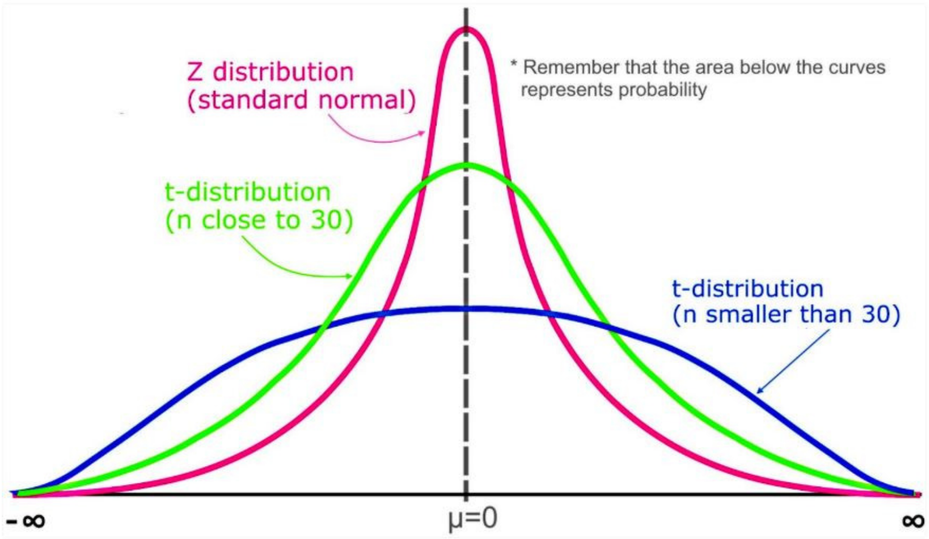
Collect n samples from the population

From these samples estimate

- The mean
- The variance

The result is a *Student t Distribution*

- Very similar to a Normal distribution
- Takes sample size into account



Normal vs. Student t Distribution

A Normal distribution is defined by two parameters

$$\mu = \frac{1}{n}\sum x_i \qquad \textit{mean}$$

$$\sigma^2 = \frac{1}{n}\sum (x_i - \mu)^2 \qquad \textit{variance}$$

Probabilities are computed using a z-score

$$z = \left(\frac{x-\mu}{\sigma}\right)$$

A Student t-Distribution is defined by three parameters

$$\bar{x} = \frac{1}{n}\sum x_i \qquad \textit{mean}$$

$$s^2 = \frac{1}{n-1}\sum (x_i - \bar{x})^2 \qquad \textit{variance}$$

$$dof = n - 1 \qquad \textit{degrees of freedom}$$

Probabilities are computed using a t-score

$$t = \left(\frac{x-\bar{x}}{s}\right)$$

t-Table

Left side

- degrees of freedom
- sample size minus 1

Top

- Area of the tail

Middle

- t-score
- similar to a z-score

Note

- t-score is equal to the z-score for infinite dof

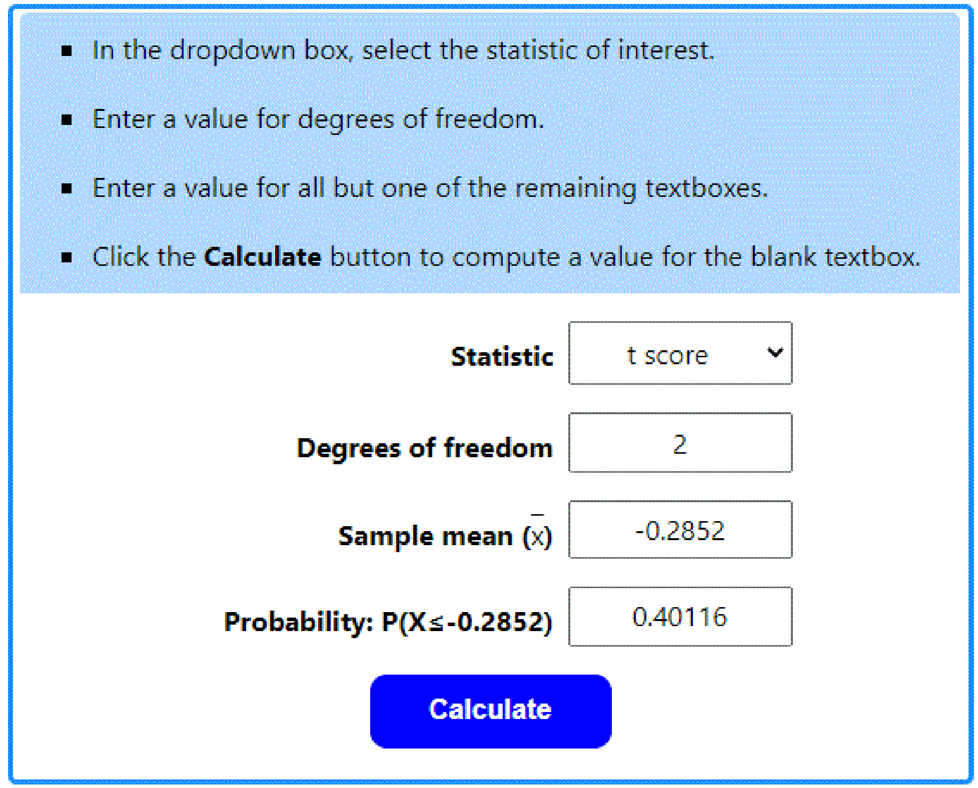
df \ p	0.001	0.0025	0.005	0.01000	0.025	0.05000
1	636.61900	318.30900	63.65670	31.82050	12.70620	6.31380
2	31.59910	22.32710	9.92480	6.96460	4.30270	2.92000
3	12.92400	10.21450	5.84090	4.54070	3.18240	2.35340
4	8.61030	7.17320	4.60410	3.74690	2.77640	2.13180
5	6.86880	5.89340	4.03210	3.36490	2.57060	2.01500
6	5.95880	5.20760	3.70740	3.14270	2.44690	1.94320
7	5.40790	4.78530	3.49950	2.99800	2.36460	1.89460
8	5.04130	4.50080	3.35540	2.89650	2.30600	1.85950
9	4.78090	4.29680	3.24980	2.82140	2.26220	1.83310
10	4.58690	4.14370	3.16930	2.76380	2.22810	1.81250
100	3.39050	3.17370	2.62590	2.36420	1.98400	1.66020

t-Table (StatTrek)

You can also use StatTrek for t Tables

- Input degrees of freedom
- Input eith the t-score or
- The probability
- Press Calculate

StatTrek computes and displays the remaining term



Note on t Tables

A sample size of 1 is meaningless

- Zero degrees of freedom
- You can't estimate two numbers (mean and variance) from a single sample

A sample size of 2 works

- t-scores are very large
- Reflects uncertainty with only 2 measurements

More samples helps

- t-score gets smaller
- Diminishing returns

df \ p	0.001	0.0025	0.005	0.01000	0.025	0.05000
1	636.61900	318.30900	63.65670	31.82050	12.70620	6.31380
2	31.59910	22.32710	9.92480	6.96460	4.30270	2.92000
3	12.92400	10.21450	5.84090	4.54070	3.18240	2.35340
4	8.61030	7.17320	4.60410	3.74690	2.77640	2.13180
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Dice: Sample Size = 1

Let's go back to rolling dice

$$Y = 4d4 + 3d6 + 2d8$$

Determine the

- Probability that y = 35,
- Probability that y > 35, and
- 90% confidence interval

using a single die roll

```
n = 1;
Roll = zeros(n,1);
for i=1:1
   d4 = ceil( 4*rand(1,4) );
   d6 = ceil( 6*rand(1,3) );
   d8 = ceil( 8*rand(1,2) );
   Y = sum(d4) + sum(d6) + sum(d8);
   Roll(i) = Y;
   end

Roll = 30
```

t-Test Calculations

To use a t-Test, you first compute the mean and variance:

$$\bar{x} = \tfrac{1}{n} \sum x_i = 30$$

$$s^2 = \frac{1}{n-1} \sum (x - \bar{x})^2 = \frac{0}{0}$$

With a sample size of one, the varianec is undefined. This tells you:

You cannot determine probabilities using a single measurement.

You need at least two measurements to do any analysis.

t Test with Sample Size of 3

Now you *can* get results

```
Roll = 30    40    28

x = 32.6667

s = 6.4291

n = 3
```

```
n = 3;
Roll = zeros(n,1);
for i=1:1
   d4 = ceil( 4*rand(1,4) );
   d6 = ceil( 6*rand(1,3) );
   d8 = ceil( 8*rand(1,2) );
   Y = sum(d4) + sum(d6) + sum(d8);
   Roll(i) = Y;
   end

x = mean(Roll)
s = std(Roll)
n = length(Roll)
```

t Test: p(y = 35)

Calculate the t-scores

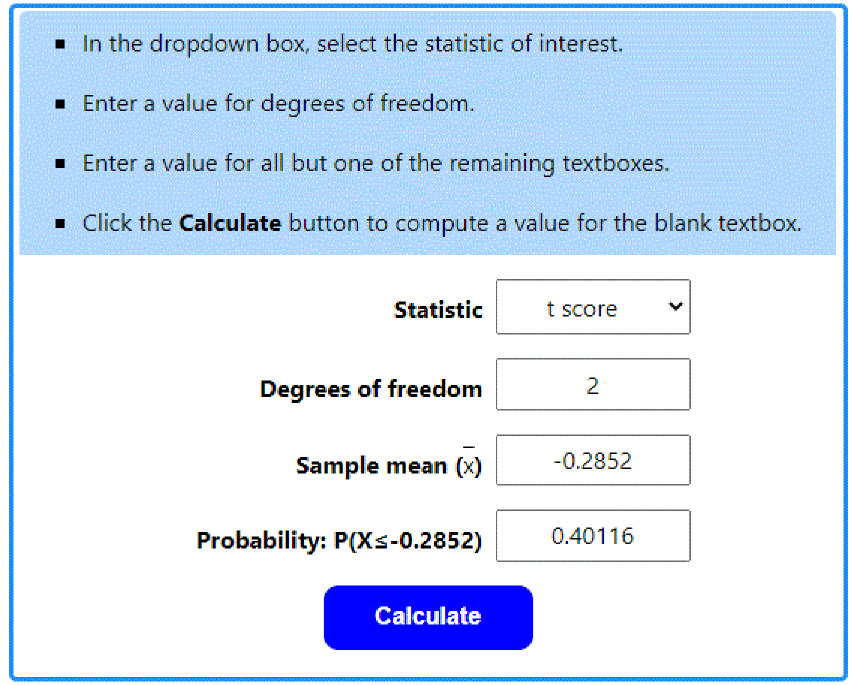
Use a t Table to convert to probabilities

```
>> t1 = (34.5 - x) / s
t1 =     0.2852

>> t2 = (35.5 - x) / s
t2 =     0.4407
```

From StatTrek, with 2 dof

- Degrees of Freedom = 2 (Sample Size - 1)
- p1 = 40.116%
- p2 = 35.124%



Difference = 4.992%

- vs. 4.4445% (exact)

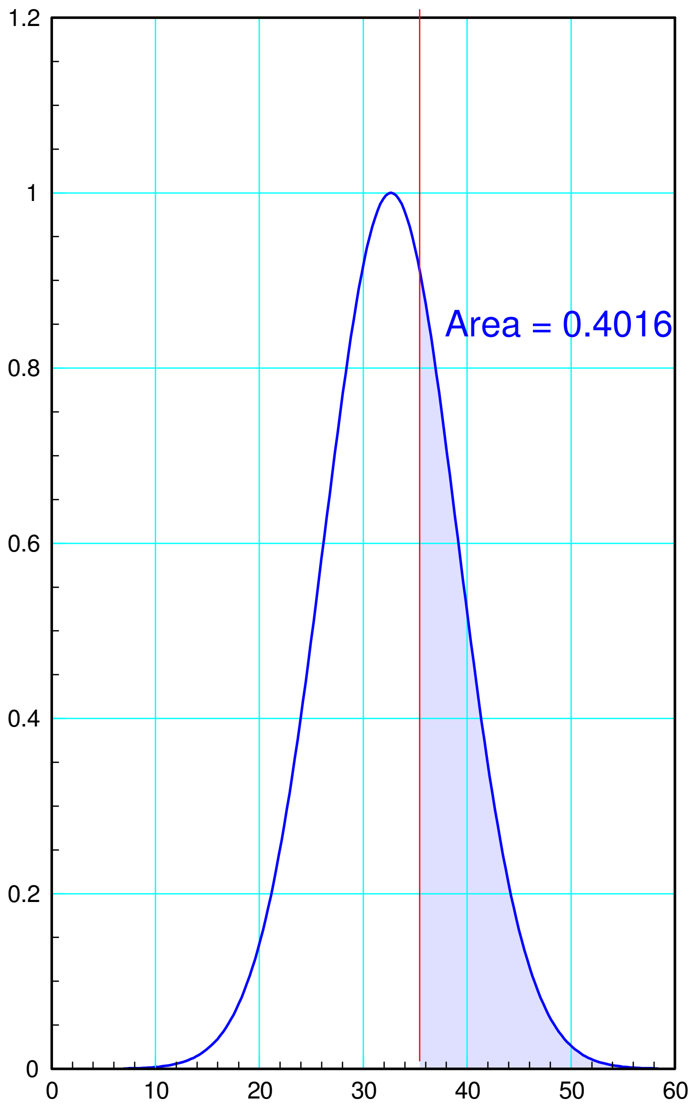
t-test: p(y>34.5)

Calculate the distance to the mean

```
>> t1 = (34.5 - x) / s
t1 =    0.2852
```

Convert this to a probability using a Student-t table

- StatTrek also works
- Degrees of Freedom = 2 (Sample Size - 1)
- p1 = 40.116%
- exact = 15.8524%



t Test: 90% Confidence Interval

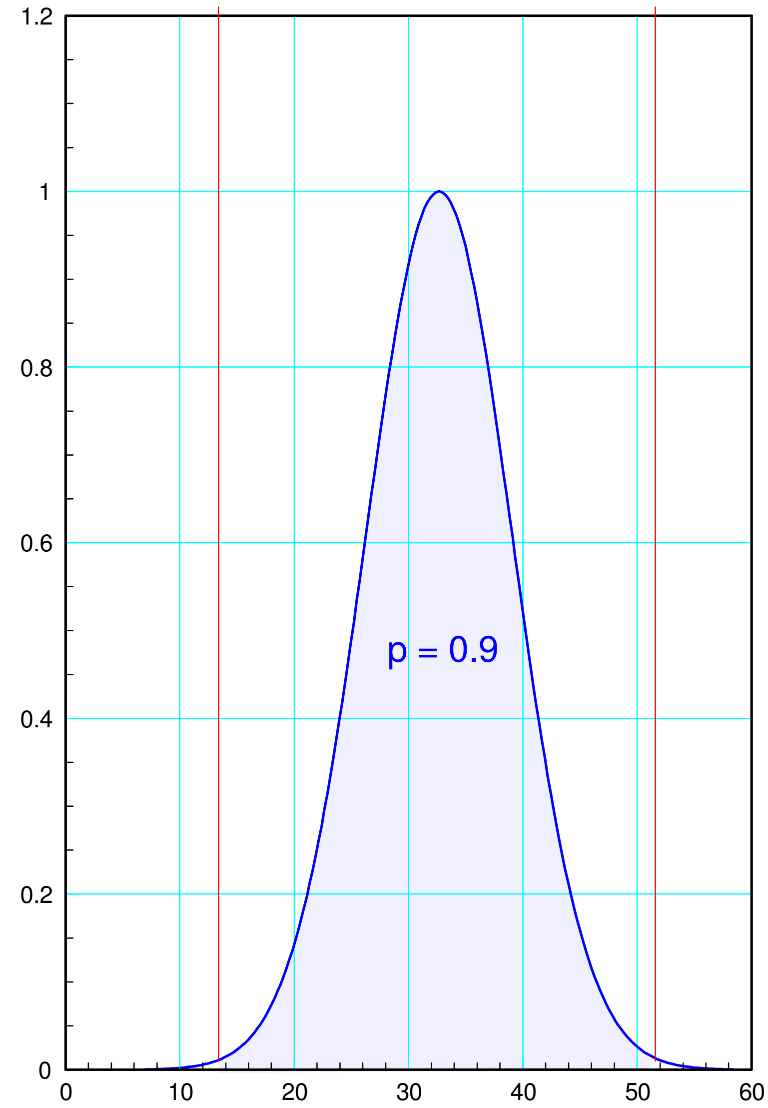
Determine the t-score for

- 2 degrees of freedom
- 5% tails
- t = 2.9200

Go left and right 2.92 standard deviations

Result

- 13.89 < y < 51.40 (p = 90%)
- 21.5 < y < 38.5 (from enumeration)



Results

Similar results to Monte Carlo

- 3 rolls vs. 1,000,000

	Monte-Carlo	Normal Approx	t-Test
p(y = 35)	4.43%	4.34%	4.992%
p(y > 35)	11.39%	11.15%	40.11%
90% confidnce interval	22 < y < 37	21.4 < y < 37.6	13.89 < y < 41.54
# rolls	1,000,000	0	3

Results get better with larger sample size

	# Rolls	p(y = 35)	p(y >= 35)	90% conf interval
Enumeration (exact)	3,538,944	4.4445%	15.8524%	(21.5, 38.5)
Monte-Carlo	100,000	4.444%	15.859%	(21.5, 38.5)
Normal Approx	0	4.344%	15.498%	(21.4, 37.6)
t-Test	3	4.992%	40.116%	(13.9, 41.5)
t-Test	10	3.75%	18.057%	(18.2, 39.5)
t-Test	30	4.27%	14.17%	(22.1, 37.2)

Populations vs. Individuals

There's a *slight* difference when asking questions about individuals vs. populations

Individuals:

$$s^2 = \left(\frac{1}{n-1}\right)\sum (x_i - \bar{x})^2$$

Matlab

```
v = var(Data)
```

Populations:

$$s^2 = \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)\sum (x_i - \bar{x})^2$$

```
v = var(Data) / n
```

If you want to know the variation of a single roll, use the formula for individuals

If you want to know the value of the population's mean, use the population formula

- You know more about populations than individuals
- As sample size goes to infinity, you know the population's mean *exactly*
- Individuals still have variability

What is the mean of y?

- Y = 4d4 + 3d6 + 2d8
- Population question
- Divide the variance by n

Sample Size	t-score (5% tails)	Population's Mean
1	-	undefined
3	2.9200	23.4718 < mean < 44.5282
10	1.83110	25.4239 < mean < 31.9761
100	1.66039	28.1737 < mean < 29.8263
1,000	1.64838	29.2831 < mean < 29.8029
infinite	1.64485	29.5000 < mean < 29.5000 *exact*

Summary

If you want to know what's coming off the assembly line, you need data.

If you measure everything (enumeration),

- You know what you're producing, but
- You have no product (nothing is new)

With a Monte-Carlo simulation,

- You get good results,
- But you need a large sample size (can be expensive)

If you know the mean and standard deviation...

- Use a Normal approximation
- Requires zero measurements

If you don't know the mean and standard deviation...

- Use a Student-t Test
- Requires a small sample size ($n \geq 2$)
- More data helps, but you don't need a huge amount of data