Chi-Squared Test

ECE 341: Random Processes

Lecture #29

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

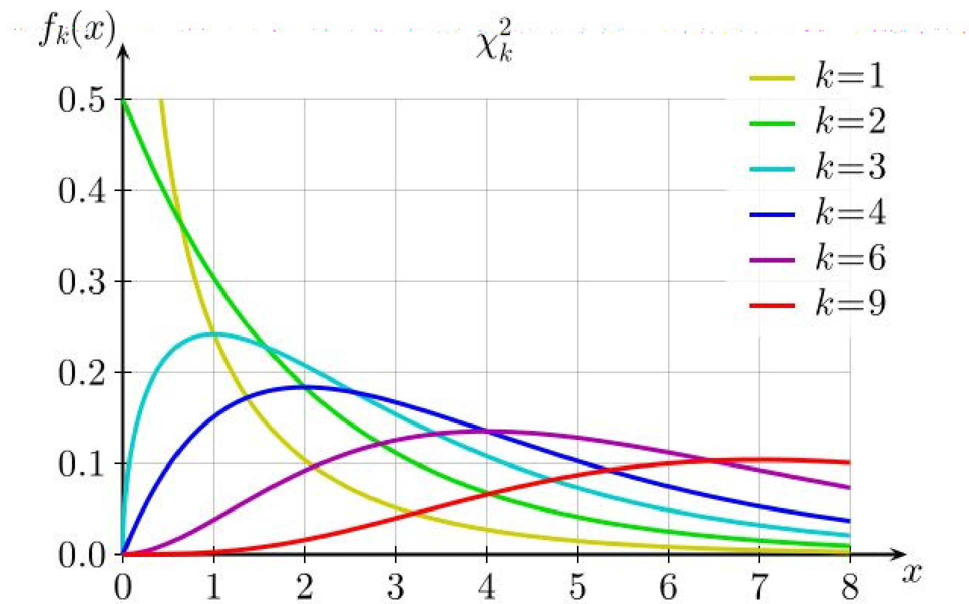
Chi-Squared Test

Is the data consistent with an assumed distribution. It is used to test

- Whether a die is fair (each number has equal probability)
- Whether a distribution is Normal (vs. Poisson or geometric)

The Chi-Squared distribution is a type of Gamma distribution:

$$(x_i - \mu)^2$$



Example: Is this a fair die?!

```
d6 = ceil(rand*6);
```

Procedure for a Chi-Squared test:

- Define M bins (6 bins in this case: numbers 1 .. 6)
- Collect n data points.
- Count how many times the data fell into each bin
- Compute the Chi-Squared total for each bin as

$$\chi^2 = \left(\frac{(np-N)^2}{np}\right)$$

  - *np: expected frequency*
  - *N: measured frequency*
- Convert to a probability using a chi-squared table
  - *degrees of freedom = # bins minus one*

Example: n = 120 die rolls

```
RESULT = zeros(1,6);

for i=1:120
    d6 = ceil(rand*6);
    RESULT(d6) = RESULT(d6) + 1;
end
```

Die Roll (bin)	p theoretical probability	np expected frequency	N actual frequency	$\chi^2 = \left(\frac{(np-N)^2}{np}\right)$
1	1/6	20	18	0.2
2	1/6	20	27	2.45
3	1/6	20	25	1.25
4	1/6	20	19	0.05
5	1/6	20	14	1.8
6	1/6	20	17	0.45
			Total:	**6.2**

Use a Chi-squared table to convert this to a probability

- Degrees of freedom = # bins - 1

Chi-Squared Table
Probability of rejecting the null hypothesis

df	99.5%	99%	97.5%	95%	90%	10%	5%	2.5%	1%	0.5%
1	7.88	6.64	5.02	3.84	2.71	0.02	0	0	0	0
2	10.6	9.21	7.38	5.99	4.61	0.21	0.1	0.05	0.02	0.01
3	12.84	11.35	9.35	7.82	6.25	0.58	0.35	0.22	0.12	0.07
4	14.86	13.28	11.14	9.49	7.78	1.06	0.71	0.48	0.3	0.21
5	16.75	15.09	12.83	11.07	9.24	1.61	1.15	0.83	0.55	0.41

Chi-Squared table. With 5 degrees of freedom (df) and $\chi^2 = 6.2$, the probability is between 90% and 10%

StatTrek.com:

- Also works (easier too)

- Enter a value for degrees of freedom.
- Enter a value for one, and only one, of the remaining unshaded text boxes.
- Click the **Calculate** button to compute values for the other text boxes.

Degrees of freedom	5
Chi-square critical value (CV)	6.2
$P(X^2 < 6.2)$	0.71
$P(X^2 > 6.2)$	0.29

The probability the die is loaded is 0.71 (www.StatTrek.com)

Repeat with 1,000,000 rolls of the dice:

```
RESULT = zeros(1,6);

n = 1e6;
p = 1/6;
for i=1:n
    d6 = ceil(rand*6);
    RESULT(d6) = RESULT(d6) + 1;
end

RESULT =   166220   166399   166933   167052   166500   166896

Chi2 = sum( (RESULT - n*p).^2) / (n*p)

Chi2 =    3.4257
```

This corresponds to a probability of 37%

- Not too large
- Not too small

Example 2: Loaded Die

Suppose instead you had a loaded die:

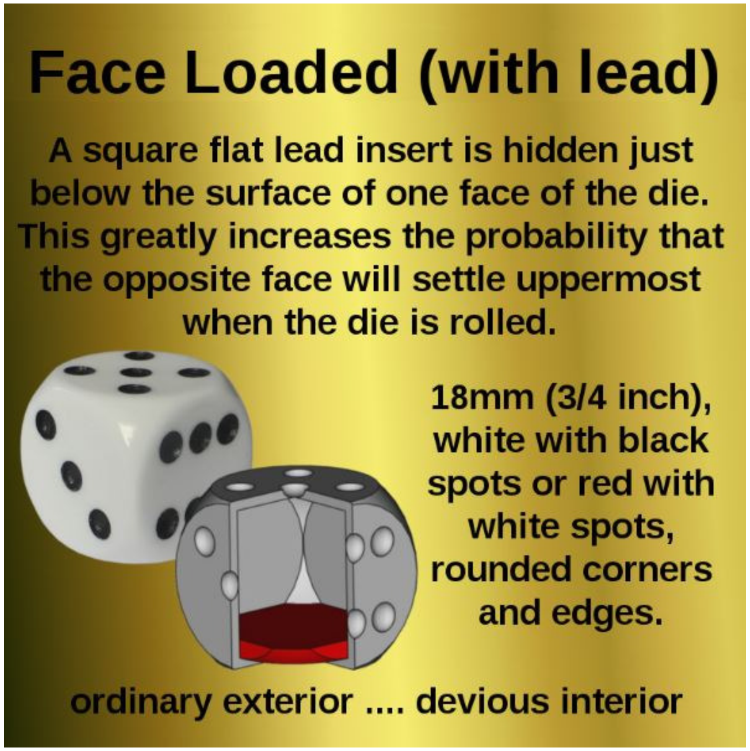
- 90% of the time, the die is fair
  - *all results have equal probability*
- 10% of the time, the result is always a 6.

Can you detect this is loaded with 120 rolls?

```
n=120;

for i=1:n
    if(rand < 0.1)
        d6 = 6;
    else
       d6 = ceil(rand*6);
    end
    RESULT(d6) = RESULT(d6) + 1;
end

RESULT =     24     18     10     17     21     30
```



Loaded Die (cont'd)

Compute the Chi-Squared score:

$$\chi^2 = 11.5$$

Die Roll (bin)	p theoretical probability	np expected frequency	N actual frequency	$\chi^2 = \left(\frac{(np-N)^2}{np}\right)$
1	1/6	20	24	0.8
2	1/6	20	18	0.2
3	1/6	20	10	5
4	1/6	20	17	0.45
5	1/6	20	21	0.05
6	1/6	20	30	5
			Total:	11.5

Loaded Die (cont'd)

Convert this to a probabity with a Chi-Squared table

- $\chi^2 = 11.5$
- 5 degrees of freedom (6 bins)
- p = 0.96 (from StatTrek)

Chi-Squared Table
Probability of rejecting the null hypothesis

df	99.5%	99%	97.5%	95%	90%	10%	5%	2.5%	1%	0.5%
1	7.88	6.64	5.02	3.84	2.71	0.02	0	0	0	0
2	10.6	9.21	7.38	5.99	4.61	0.21	0.1	0.05	0.02	0.01
3	12.84	11.35	9.35	7.82	6.25	0.58	0.35	0.22	0.12	0.07
4	14.86	13.28	11.14	9.49	7.78	1.06	0.71	0.48	0.3	0.21
5	16.75	15.09	12.83	11.07	9.24	1.61	1.15	0.83	0.55	0.41

Chi-Squared table. With 5 degrees of freedom (df) and χ^2=11.5, the probability is about 95%

Loaded Die with 1200 Rolls

```
RESULT = zeros(1,6);
N = 1200;

for i=1:N
    if(rand < 0.1)
        d6 = 6;
    else
       d6 = ceil(rand*6);
    end
    RESULT(d6) = RESULT(d6) + 1;
end

RESULT =   186   180   160   173   197   304

Chi2 = sum( (RESULT - N*p) .^ 2 ) / (N*p)

Chi2 =   68.7500
```

With 1200 rolls, I'm > 99.95% certain this die is loaded

- and I'm probably broke after 1200 rolls of a loaded die

Example 3: How loaded is too loaded?

- Load a die
- p(detection) = 5% after 120 rolls.

Solution: χ^2 score = 11.1

- Assume x too many 6's rolled gives χ^2 = 11.2
- Assume same number of rolls for all other numbers

How Loaded is Too Loaded?

- You can get away with 13.84 extra sixes
- p(loading) = 13.84 / 120 = 11.5%

Die Roll (bin)	p theoretical probability	np expected frequency	N actual frequency	$\chi^2 = \left(\frac{(np-N)^2}{np}\right)$
1	1/6	20	20 - x/5	$\left(\frac{(x/5)^2}{20}\right)$
2	1/6	20	20 - x/5	$\left(\frac{(x/5)^2}{20}\right)$
3	1/6	20	20 - x/5	$\left(\frac{(x/5)^2}{20}\right)$
4	1/6	20	20 - x/5	$\left(\frac{(x/5)^2}{20}\right)$
5	1/6	20	20 - x/5	$\left(\frac{(x/5)^2}{20}\right)$
6	1/6	20	20 + x	$\left(\frac{x^2}{20}\right)$
			Total:	$\left(\frac{1.2x^2}{20}\right) = 11.5$

The Imitation Game

- Example of "How Loaded is Too Loaded"

Back in WWII, Alan Turing broke the German Enigma code

- Movie: *The Imitation Game*

Problem:

- How many times can you respond to German messages
- Without the Germans realizing their code was cracked?

Chi-Squared Problem

Fudging Data

Chi-Squared tests can also detect if data was fudged.

- Large χ^2 means the data doesn't match the assumed distribution
- Small χ^2 means the data fits the assumed discription too well (data was forged)
  - *It's possible but unlikely to get such good data*

Chi-Squared Table

Probability of rejecting the null hypothesis

df	99.5%	99%	97.5%	95%	90%	10%	5%	2.5%	1%	0.5%
1	7.88	6.64	5.02	3.84	2.71	0.02	0	0	0	0
2	10.6	9.21	7.38	5.99	4.61	0.21	0.1	0.05	0.02	0.01
3	12.84	11.35	9.35	7.82	6.25	0.58	0.35	0.22	0.12	0.07
4	14.86	13.28	11.14	9.49	7.78	1.06	0.71	0.48	0.3	0.21
5	16.75	15.09	12.83	11.07	9.24	1.61	1.15	0.83	0.55	0.41

Fudging Data

You can spot fake data with a Chi-Squared test

Example: Rolling Dice

- Only roll a die 100 times, and then
- Add 150 to the sum of each die roll
- Claim I *actually* rolled the dice 1000 times.

This fake data shows up with a chi-squared test

- Data is too good

Matlab Code

```
Result = 150*ones(1,6);

for i=1:100
   n = ceil(6*rand);
   Result[n] = Result[n] + 1;
   n

disp(Result)

168  168  159  169  169  167
```

Fudging Data (cont'd)

Determine the χ^2 score (0.44)

- p = 0.5% (from StatTrek)
- The odds against getting such good data are 200 : 1 against.
- Most likely the data was faked.

Die Roll (bin)	p theoretical	np expected #	N actual #	$\chi^2 = \left(\frac{(np-N)^2}{np}\right)$
1	1/6	166.67	168	0.0106
2	1/6	166.67	168	0.0106
3	1/6	166.67	159	0.353
4	1/6	166.67	169	0.0326
5	1/6	166.67	169	0.0326
6	1/6	166.67	167	0.0007
			Total:	0.44

Fudging Data (take 2)

Rather than generate random numbers, go through a pseudo-random sequence

- Rolls: 5, 2, 6, 3, 4, 1, repeat

This shows up in a chi-squared test

- Chi-Squared = 0.000
- The data is too good
- It isn't random

Matlab Code

```
Table = [5,2,6,3,4,1];

for i=1:120
   n = mod(i,6);
   Die = Table(n);
   Result[Die]=Result[Die]+1;
   end


disp(Result)

20  20  20  20  20  20
```

Fudging Data (take 3)

- Mendel's Experiment

Mendel came up with idea of

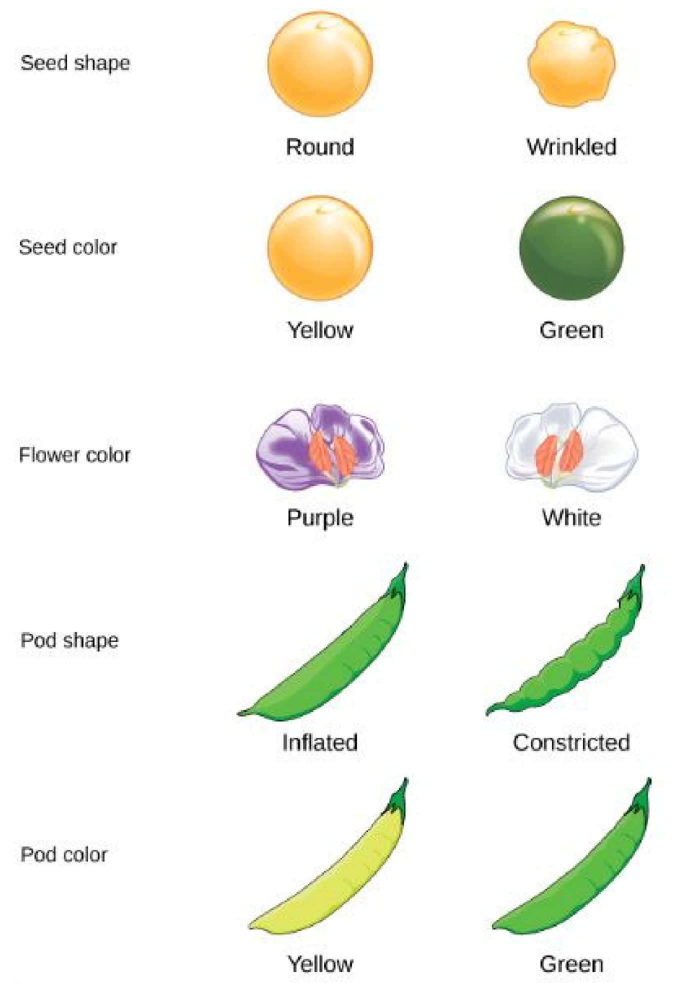
- Dominant genes and
- Recessive genes

If either gene is dominant

- the dominant trait is expressed

If both genes are recessive

- The recessive trait is expressed



Mendel's Experiment

Start with two pure-breds

- 1st generation

2nd generation will be

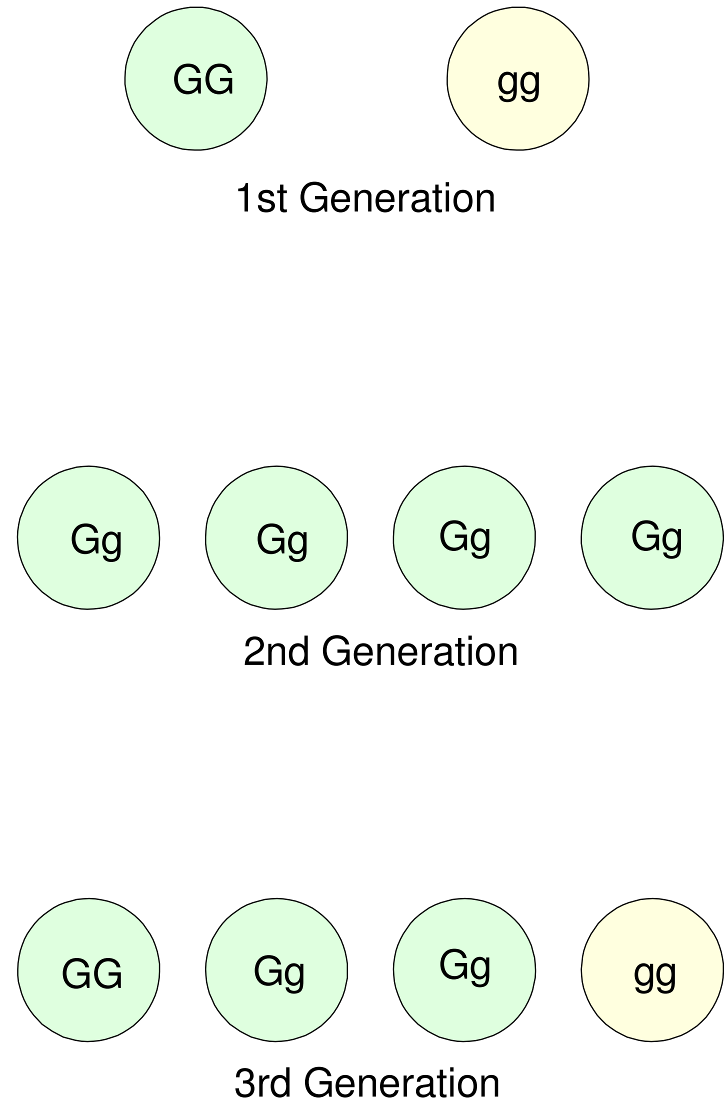
- 100% dominant trait

3rd generation will be

- 75% dominant trait
- 25% recessive trait

The results will not be *exactly* 75% : 25%

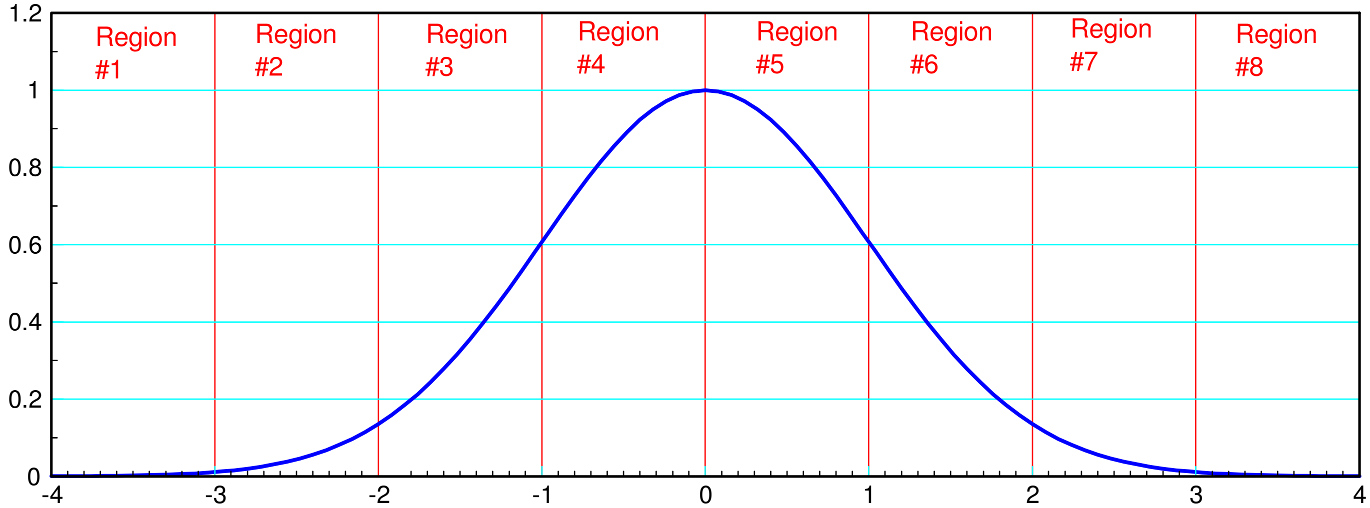
- This is a random process
- Yet Mendel's results were *always* 75% : 25%
- There were other problems too...



Chi-Squared with Continuous Distributions

Also works with continuous distributions

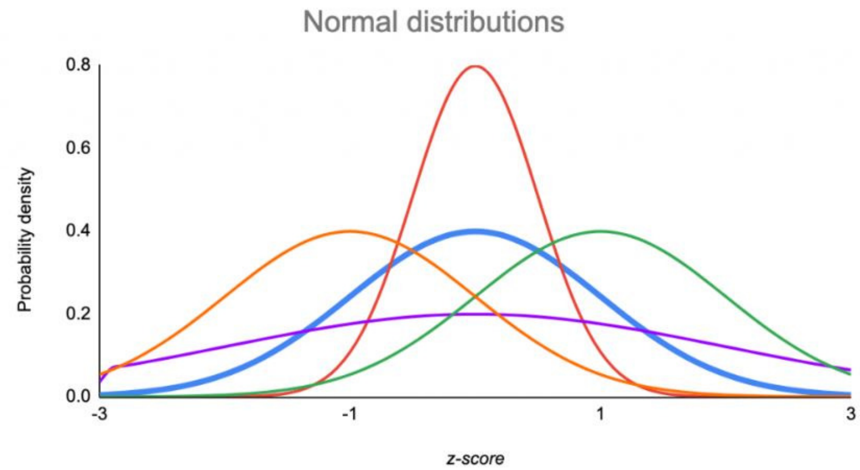
- Split the continuous variable into N distinct regions (many ways to do this)
- Calculate the probability that any given data point will fall into each region (p)
- Calculate the expected number of observations you should have in each region (np),
- Compare to the actual frequency (N), i.e. calculate the χ^2 score, then
- Convert the chi-squared score into a probability.



Normal Distribution

Example:

```
X = sum( rand(12,1) ) - 6
```
$\approx N(0, 1)$



Is this a standard Normal distribution?

- No - I can see the code

Can you detect that it is *not* a standard normal variable?

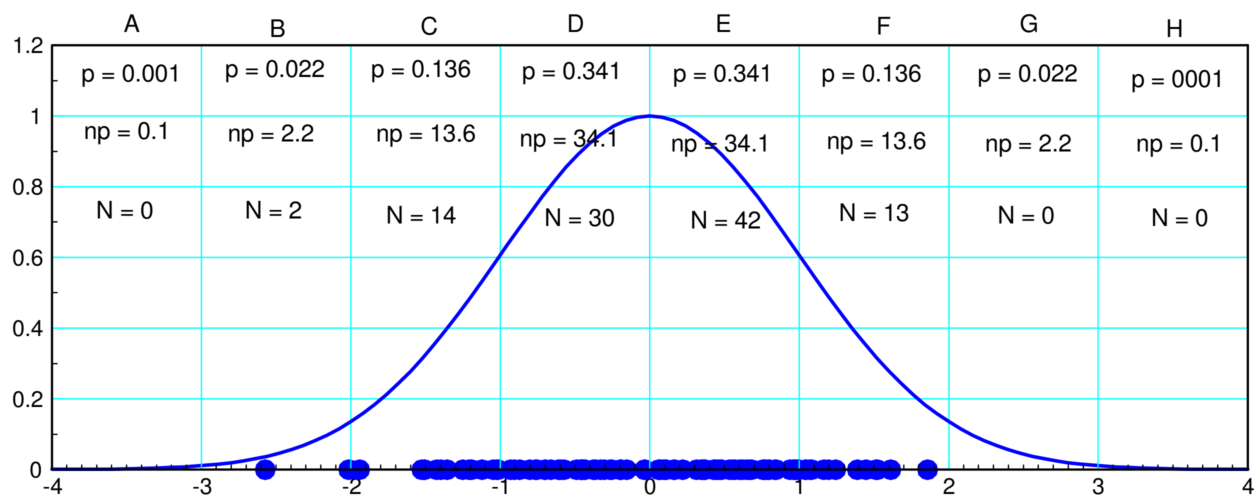
- harder.
- Use a χ^2 table

Example: Generate 100 random numbers

```
X = [];

for i=1:100
   X = [X ; sum( rand(12,1) ) - 6];
   end
```

- Split the X axis into 8 regions (A..H) (this is somewhat arbitrary).
- Compute the probability of each region (p) and the expected frequency (np)
- Count how many times X fell into each region (N)
- From this, create a Chi-Squared table



Compute the χ^2 score

- $\chi^2 = 4.7906$
- p = 0.31 (from StatTrek)
- Can't tell with only 100 data points

Region (bin)	p	np	N	$\chi^2 = \left(\frac{(np-N)^2}{np}\right)$
A	0.001	0.1	0	0.1
B	0.022	2.2	2	0.0182
C	0.138	13.8	14	0.0029
D	0.341	34.1	30	0.493
E	0.341	34.1	42	1.8302
F	0.138	13.8	13	0.0464
G	0.022	2.2	0	2.2
H	0.001	0.1	0	0.1
			Total:	**4.7906**

Repeat with 100,000 random numbers.

- $\chi^2 = 36.9886$
- p > 0.9995 (from StatTrek)
- With enough data I can tell that this isn't really a standard normal distribution
- The information is in the tails.

Region	p	np	N	$\chi^2 = \left(\frac{(np-N)^2}{np}\right)$
A	0.0013	132	97	9.2803
B	0.0214	2,140	2,085	1.4136
C	0.1359	13,591	13,751	1.8836
D	0.3413	34,134	34,067	0.1315
E	0.3413	34,134	33,845	2.4469
F	0.1359	13,591	13,895	6.7998
G	0.0214	2,140	2,168	0.3664
H	0.0013	132	88	14.6667
			Total:	**36.9886**

Summary

A chi-squared test is a test of a distribution

- Split the data in the N bins
- Compare the expected frequency to the observed frequency

With it, you can detect

- If a die is loaded
- If data was fudged