ECE 376 - Homework #11

z-Transforms and Digital Filters.   Due Monday, November 20th
Please email to jacob.glower@ndsu.edu, or submit as a hard copy, or submit on BlackBoard

1) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{3s+6}{s^2+10s+30}\right)X$$

a) What is the differential equation relating X and Y?

Cross multiply

$$(s^2 + 10s + 30)Y = (3s + 6)X$$

Note that sY means *the derivative of y*

$$\boxed{y'' + 10y' + 30y = 3x' + 6x}$$

b) Find y(t) assuming

$$x(t) = 6 + 5\sin(4t)$$

Use phasors and superposition:

x(t) = 6

$$s = 0$$

$$X = 6$$

$$Y = \left(\frac{3s+6}{s^2+10s+30}\right)_{s=0} \cdot (6)$$

$$Y = 1.20$$

$$y(t) = 1.20$$

x(t) = 5 sin(4t)

$$s = j4$$

$$X = 0 - j5$$

real = cosine, -imag = sine

$$Y = \left(\frac{3s+6}{s^2+10s+30}\right)_{s=j4} \cdot (0 - j5)$$

$$Y = -0.2004 - j1.5702$$

$$y(t) = -0.2004\cos(4t) + 1.5702\sin(4t)$$

The total answer is DC + AC

$$\boxed{y(t) = 1.20 - 0.2004\cos(4t) + 1.5702\sin(4t)}$$

2) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{0.1(z+1)}{(z-0.9)(z-0.8)} \right) X$$

a) What is the difference equation relating X and Y?

Cross multiply

$$(z-0.9)(z-0.8)Y = 0.1(z+1)X$$

$$(z^2 - 1.7z + 0.72)Y = 0.1(z+1)X$$

Note that zY means *y(k+1)* or *the next value of y(k)*

$$\boxed{y(k+2) - 1.7y(k+1) + 0.72y(k) = 0.1(x(k+1) + x(k))}$$

b) Find y(t) assuming a sampling rate of T = 0.01 second

$$x(t) = 6 + 5\sin(4t)$$

Use superposition and phasors

x(t) = 6

$$s = 0$$

$$z = e^{sT} = 1$$

$$Y = \left(\frac{0.1(z+1)}{(z-0.9)(z-0.8)} \right)_{z=1} \cdot (6)$$

$$Y = 60.0$$

x(t) = 5 sin(4t)

$$X = 0 - j5$$

$$s = j4$$

$$z = e^{sT} = e^{j0.04}$$

$$Y = \left(\frac{0.1(z+1)}{(z-0.9)(z-0.8)} \right)_{z=e^{j0.04}} \cdot (0 - j5)$$

$$Y = -24.4898 - j38.9487$$

meaning

$$y(t) = -24.4898\cos(4t) + 38.0487\sin(4t)$$

The total answer is DC + AC

$$\boxed{y(t) = 60.00 - 24.4898\cos(4t) + 38.0487\sin(4t)}$$

3) Assume G(s) is a low-pass filter with real poles:

$$G(s) = \left(\frac{2000}{(s+5)(s+10)(s+20)}\right)$$

3) Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

The conversion from the s-plane to the z-plane is

$$z = e^{sT}$$

Converting the three poles:

$$s = -5 \qquad z = e^{sT} = 0.9512$$

$$s = -10 \qquad z = e^{sT} = 0.9048$$

$$s = -20 \qquad z = e^{sT} = 0.8187$$

So, the form of G(z) is

$$G(z) = \left(\frac{k}{(z-0.9512)(z-0.9048)(z-0.8187)}\right)$$

To find 'k', match the DC gain (any frequency works)

$$G(s=0) = \left(\frac{2000}{(s+5)(s+10)(s+20)}\right)_{s=0} = 2.00$$

Pick 'k' to set the DC gain of G(z) to be the same

$$G(z=1) = \left(\frac{k}{(z-0.9512)(z-0.9048)(z-0.8187)}\right)_{z=1} = 2.000$$

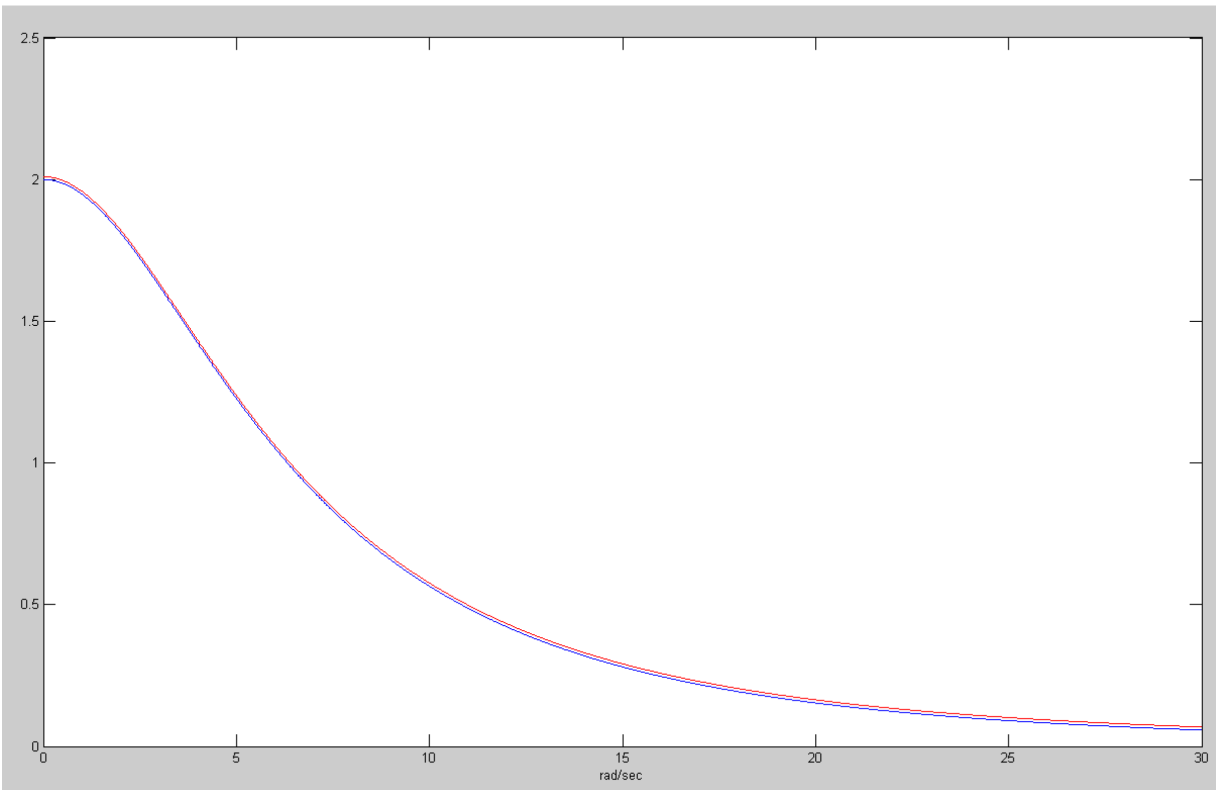
$$k = 0.001648$$

so

$$\boxed{G(z) = \left(\frac{0.0016846}{(z-0.9512)(z-0.9048)(z-0.8187)}\right)}$$

Plotting the gain vs. frequency in Matlab

```
>> w = [0:0.01:30]';
>> s = j*w;
>> Gs = 2000 ./ ( (s+5).*(s+10).*(s+20) );


>> T = 0.01;
>> z = exp(s*T);
>> Gz = 0.0016846 ./ ( (z-0.9512).*(z-0.9048).*(z-0.8187) );
>> plot(w,abs(Gs),'b',w,abs(Gz)+0.01,'r')
>> xlabel('rad/sec')
```



G(s) (blue) & G(z) (red).

Note: G(z) is offset by 0.01 so you can see the two curves

4) Assume G(s) is the following band-pass filter:

$$G(s) = \left(\frac{30s}{(s+3+j10)(s+3-j10)}\right)$$

Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

Again, convert from the s-plane to the z-plane as $z = e^{sT}$

$$s = 0 \qquad z = e^{sT} = 1$$

$$s = -3 + j10 \qquad z = e^{sT} = 0.9656 + j0.0969$$

$$s = -3 - j10 \qquad z = e^{sT} = 0.9656 - j0.0969$$

So, the form of G(z) is

$$G(z) = \left(\frac{k(z-1)}{(z-0.9656+j0.0969)(z-0.9656-j0.0969)}\right)$$

Pick k to match the gain somewhere. DC doesn't work since the gain is zero. Pick somewhere else like s = j10

$$G(s = j10) = \left(\frac{30s}{(s+3+j10)(s+3-j10)}\right)_{s=j10} = 4.8900 + j0.7335$$

$$|G(s = j10)| = 4.9447$$

Match the gain of G(z) at this frequency

$$z = e^{sT} = e^{j0.1}$$

$$\left|\left(\frac{k(z-1)}{(z-0.9656+j0.0969)(z-0.9656-j0.0969)}\right)\right|_{z=e^{j0.1}} = 4.9447$$

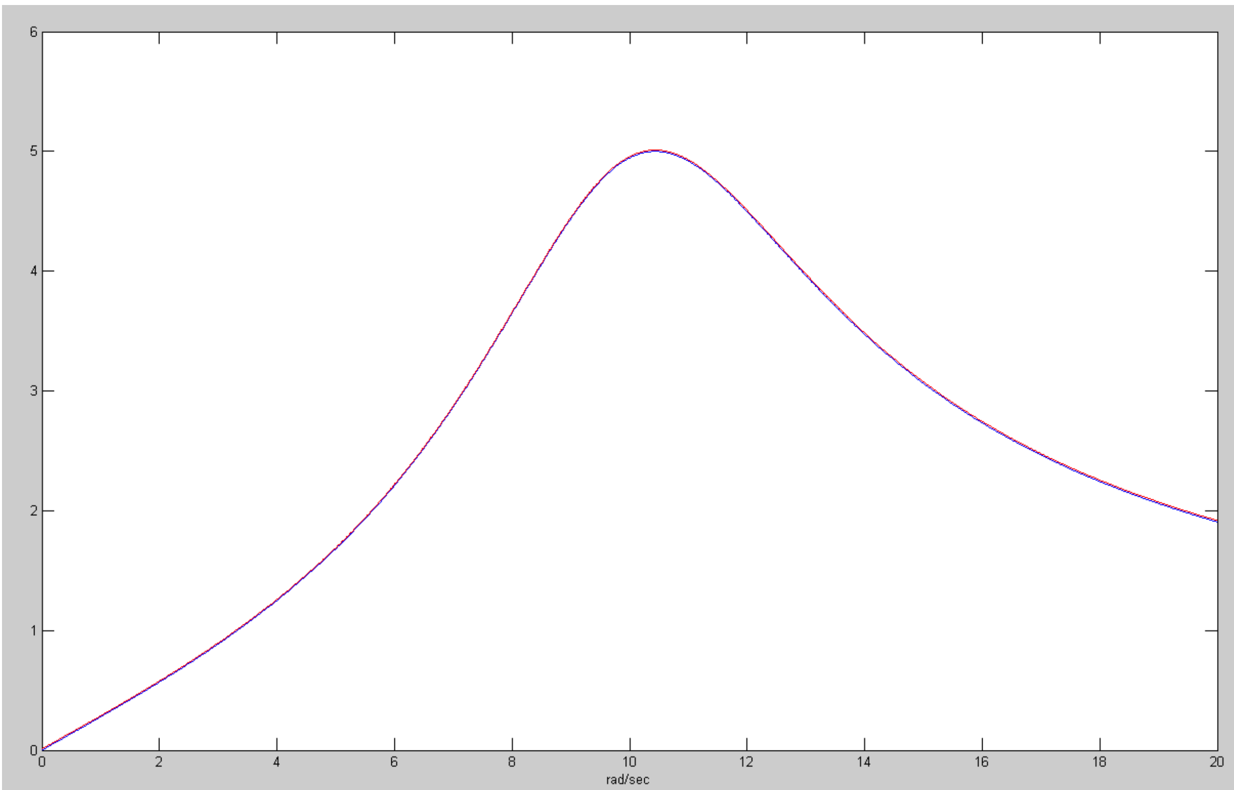
$$k = 0.2908$$

so

$$\boxed{G(z) = \left(\frac{0.2908(z-1)}{(z-0.9656+j0.0969)(z-0.9656-j0.0969)}\right)}$$

Validation: Plot the gain vs. frequency for G(s) and G(z)

```
>> w = [0:0.01:20]';
>> s = j*w;
>> Gs = 30*s ./ ( (s+3+j*10).*(s+3-j*10) );
>>
>> T = 0.01;
>> z = exp(s*T);
>> Gz = 0.2908*(z-1) ./ ( (z-0.9656+j*0.0969).*(z-0.9656-j*0.0969) );
>> plot(w,abs(Gs),'b',w,abs(Gz)+0.01,'r')
>> xlabel('rad/sec')
>>
```



G(s) (blue) & G(z) (red).
Note: G(z) is offset by 0.01 so you can see the two curves

5) Write a C program to implement the digital filter, G(z)

$$Y = \left(\frac{0.2908(z-1)}{(z-0.9656+j0.0969)(z-0.9656-j0.0969)} \right) X$$

Multiply out the polynomials and cross multiply

$$Y = \left(\frac{0.2908(z-1)}{z^2-1.9312z+0.9418} \right) X$$

$$(z^2 - 1.9312z + 0.9418)Y = 0.2908(z-1)X$$

Convert to a difference equation

y(k+2) - 1.9312 y(k+1) + 0.9418 y(k) = 0.2908 (x(k+1) - x(k))

Shift by two (change of variable)

y(k) - 1.9312 y(k-1) + 0.9418 y(k-2) = 0.2908 (x(k-1) - x(k-2))

Solve for y(k)

y(k) = 1.9312 y(k-1) - 0.9418 y(k-2) + 0.2908 (x(k-1) - x(k-2))

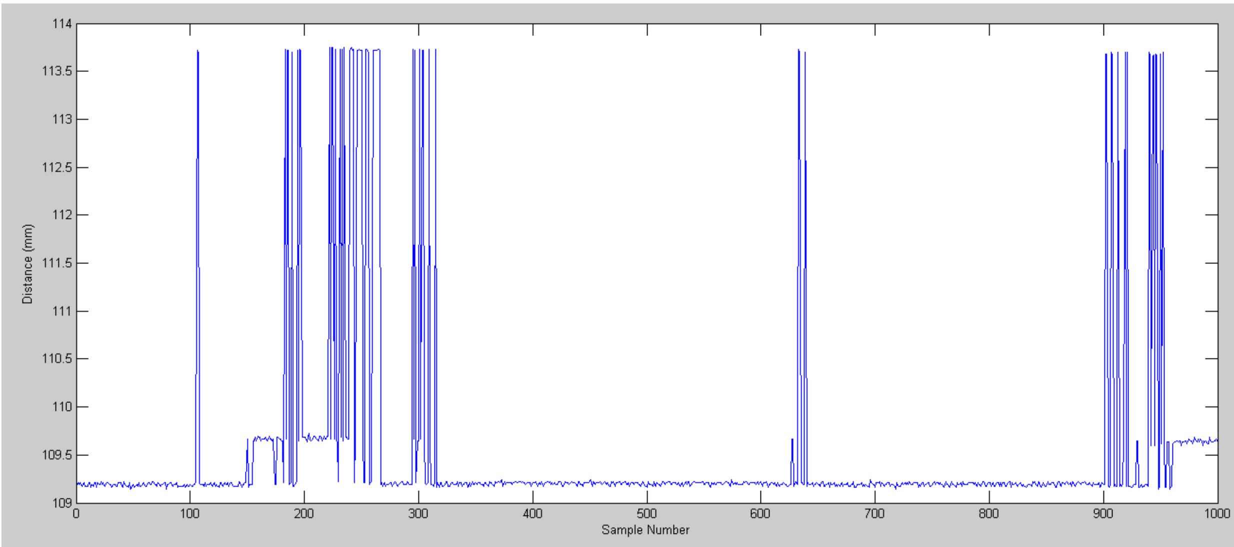
Convert to C Code

```
while(1) {

   x2 = x1;
   x1 = x0
   x0 = A2D_Read(0);

   y2 = y1;
   y1 = y0
   y0  = 1.9312*y1 - 0.9418*y2 + 0.2908 * ( x1 - x2 );

   D2A(y0);
   Wait_10ms();

   }
```

Filters & Range Measurement

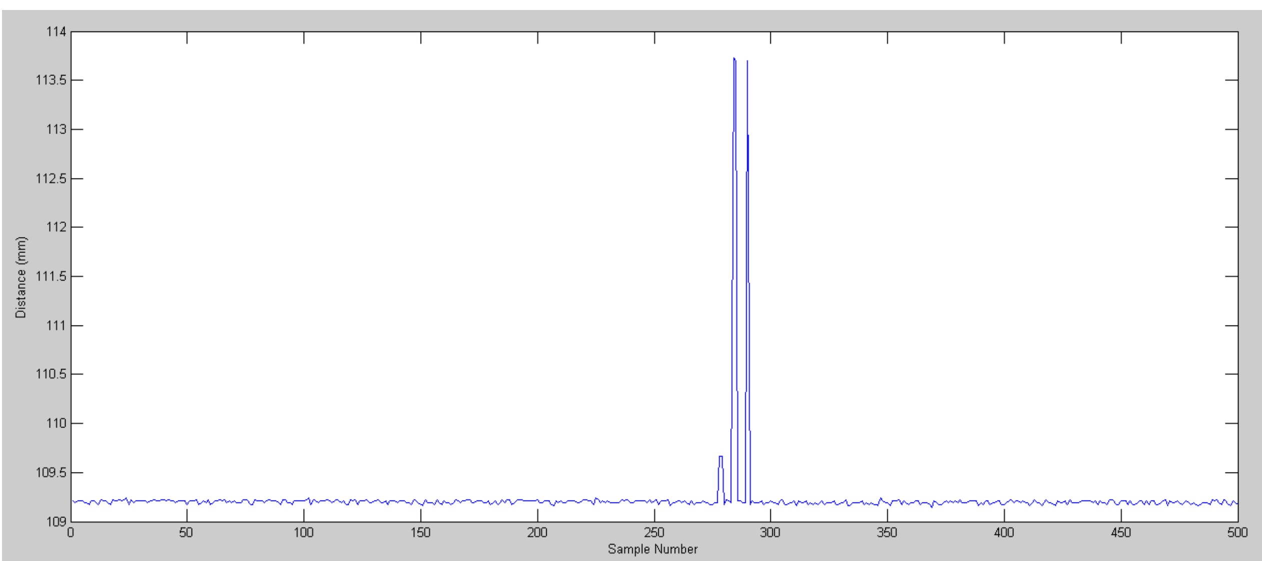
6) Collect 1000 range measurements using your ultrasonic range sensor (from homework #10).

- Plot the raw data (Matlab recommended)



Let's use data from 350 to 850 (looks like cleaner part of the data)

```
>> Data = Data(350:850);
>> k = [1:length(Data)]';
>> plot(k,Data);
>> xlabel('Sample Number');
>> ylabel('Distance (mm)');
>>
```



7) For your raw data, compute

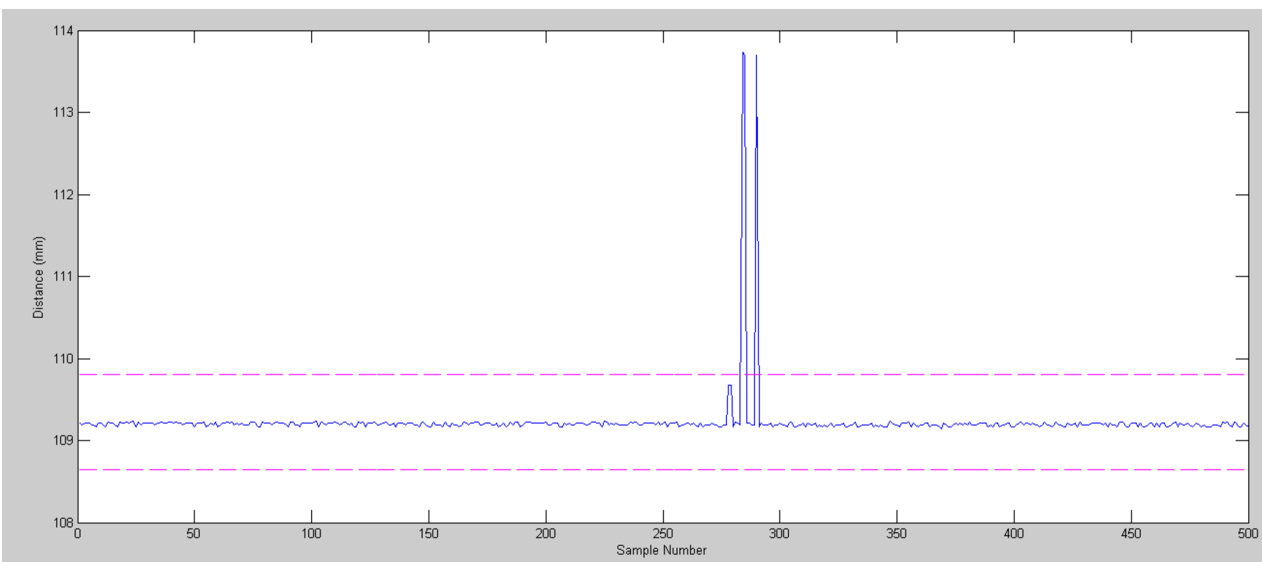
- The mean and standard devation
- The 90% confidence interval for your data.

With a sample size of 500, the t-score for 5% tails is 1.648 (StatTrek)

```
>> x = mean(Data)

x =  109.2262

>> s = std(Data)

s =    0.3504

>> x + 1.648*s

ans =  109.8037

>> x - 1.648*s

ans =  108.6487
```

90% of the range data will lie in the range of (108.6487mm - 109.8037mm)

```
>> plot(k,Data,'b',k,0*k+108.6487,'m--',k,0*k+109.8037,'m--');
>> xlabel('Sample Number');
>> ylabel('Distance (mm)');
```



Range Measurements & 90% Confidence Interval

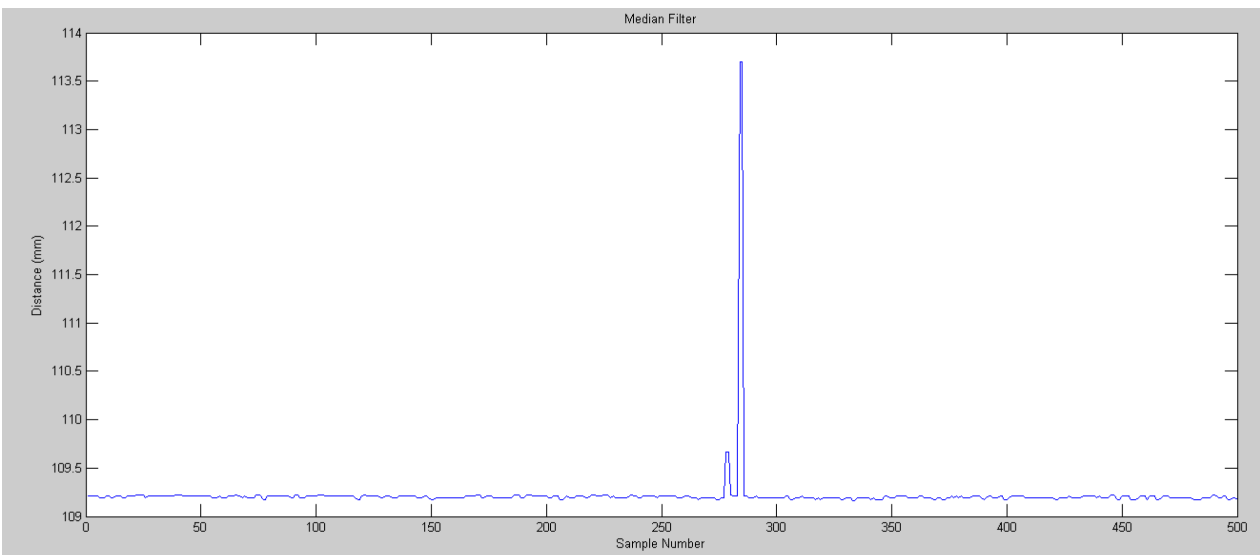
8) Filter your data with a median filter to remove glitches

For the filtered data,

- Plot the data (D2)
- Compute the mean and standard deviation of D2
- Compute the 90% confidence interval of D2

```
>> D2 = Data
for n=2:length(Data)-1
   Y = sort(Data(n-1:n+1));
   D2(n) = Y(2);
   end
k = [1:length(D2)]';
plot(k,D2);
>> xlabel('Sample Number');
>> ylabel('Distance (mm)');
>> title('Median Filter')
>>
```

Helped a little, but there are still glitches. Manually get rid of the glitchesr



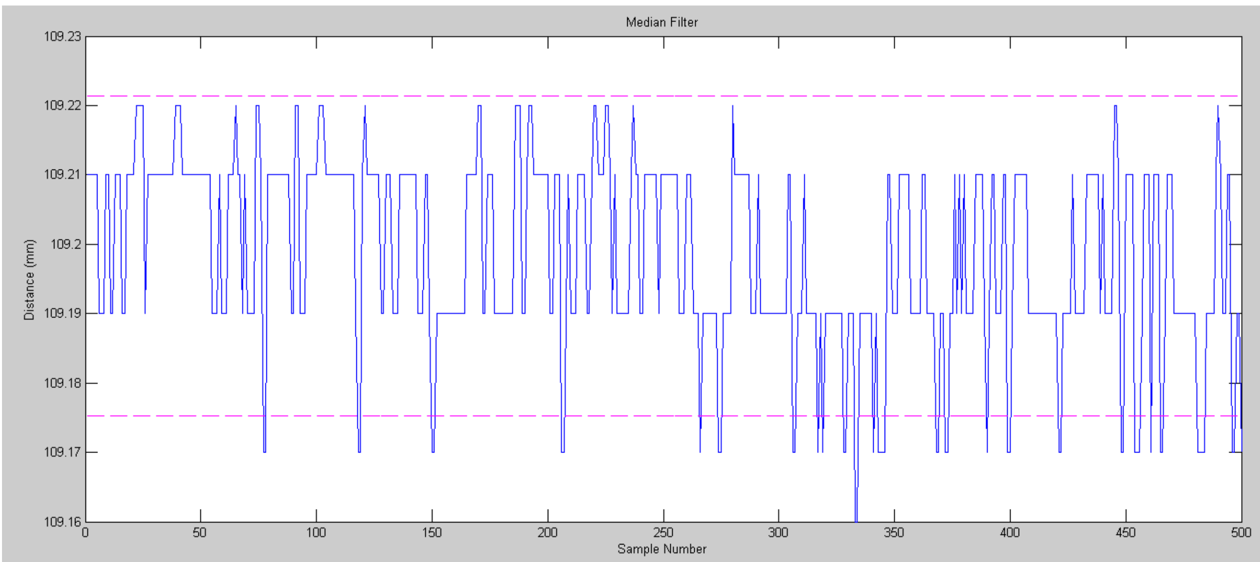
Manually get rid of the glitches

```
>> D2(278) = D2(277);
>> D2(279) = D2(277);
>> D2(284) = D2(283);
>> D2(285) = D2(283);
>> plot(k,D2);
>> xlabel('Sample Number');
>> ylabel('Distance (mm)');
>> title('Median Filter')
```

Find the mean, standard deviation, and 90% confidence interval

```
x =  109.1983

>> s = std(D2)
```

```
s =    0.0140

>> high = x + 1.648*s

high =  109.2214

>> low = x - 1.648*s

low =  109.1752

>> plot(k,D2,'b',k,0*k+high,'m--',k,0*k+low,'m--');
>> xlabel('Sample Number');
>> ylabel('Distance (mm)');
>> title('Median Filter')
```



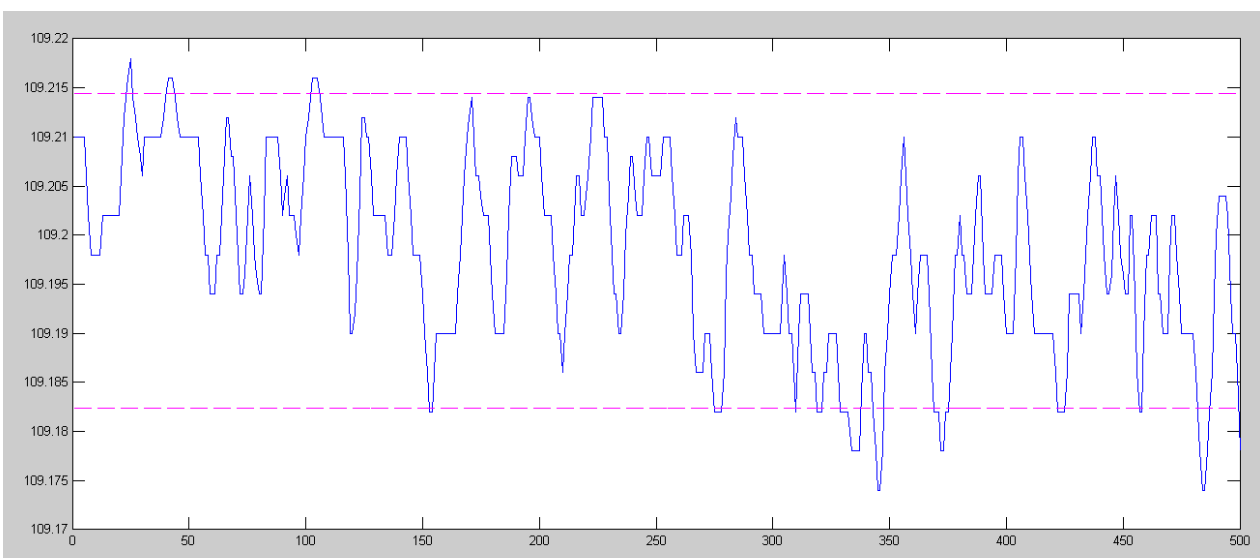
++++++++++++++++++++++9) Filter your data with a FIR filter (average of the last five data points). In Matlab:

```
D3 = D2;
for n=5:length(D2)
   D3(n) = mean(D2(n-4:n));
   end
k = [1:length(D3)]';
plot(k,D3);
```

For the filtered data,

- Plot the data (D3)
- Compute the mean and standard deviation of D3
- Compute the 90% confidence interval of D3

```
>> x = mean(D3)

x =  109.1984

>> s = std(D3)

s =    0.0097

>> high = x + 1.648*s

high =  109.2144

>> low = x - 1.648*s

low =  109.1824

>> plot(k,D3,'b',k,0*k+high,'m--',k,0*k+low,'m--');
>>
```



10) Filter your data with a IIR filter (2nd-order Butterworth low-pass filter). In Matlab:

$$Y = \left(\frac{4}{s^2+4s+4}\right)X \qquad \text{s-plane, poles at s} = -2 \pm j2$$

$$Y = \left(\frac{0.0008}{z^2-1.9600z+0.9608}\right)X \qquad \text{same filter in the z-plane with T = 10ms}$$

For the filtered data (y), determine The mean of y / The standard deviation of y / The 90% confidence interval for the next value of y. Also plot the filtered data, y(k)

```
>> x = D2;
y(1:2) = mean(x);
for k=3:length(x)
    y(k) = 1.9600*y(k-1)-0.9608*y(k-2)+0.0008*x(k-2);
    end
k = [1:length(x)]';
plot(k,y)
>> x = mean(y)

x =   109.1988

>> s = std(y)

s =    0.0051

>> high = x + 1.648*s

high =  109.2073

>> low = x - 1.648*s

low =  109.1904

>> plot(k,y,'b',k,0*k+high,'m--',k,0*k+low,'m--');
>> xlabel('Sample Number');
>> ylabel('Distance (mm)');
```