ECE 376 - Homework #11

z-Transforms and Digital Filters. Due Friday, April 21st
Please email to jacob.glower@ndsu.edu, or submit as a hard copy, or submit on BlackBoard

1) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{5(s+2)}{(s^2+4s+30)}\right)X$$

a) What is the differential equation relating X and Y?

Cross multiply

$$(s^2+4s+30)Y = 5(s+2)X$$

'sY' means 'the derivative of y'

$$y'' + 4y' + 30y = 5x' + 10x$$

b) Find y(t) assuming

$$x(t) = 6 + 5\sin(4t)$$

Use superposition

x(t) = 6

s = 0

X = 6

$$Y = \left(\frac{5(s+2)}{(s^2+4s+30)}\right)_{s=0} \cdot (6)$$

$$Y = 2$$

x(t) = 5 sin(4t)

s = j4

X = 0 - j5 *real = cosine, -imag = sine*

$$Y = \left(\frac{5(s+2)}{(s^2+4s+30)}\right)_{s=j4} \cdot (0-j5)$$

$$Y = 1.3274 - j5.0885$$

$$y(t) = 1.3274\cos(4t) + 5.0885\sin(4t)$$ *real = cosine, -imag = sine*

The total answer is DC + AC

$$y(t) = 2 + 1.3274\cos(4t) + 5.0885\sin(4t)$$

2) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{0.01(z+1)}{(z-0.96)(z-0.9)}\right)X$$

a) What is the difference equation relating X and Y?

Cross multiply

$$(z^2 - 1.86z + 0.864)Y = 0.01(z+1)X$$

note that zX means x(k+1)

$$y(k+2) - 1.86y(k+1) + 0.864y(k) = 0.01(x(k+1) + x(k))$$

or doing a time shift (change of variable)

$$y(k) - 1.86y(k-1) + 0.864y(k-2) = 0.01(x(k-1) + x(k-2))$$

either answer is valid.

b) Find y(t) assuming a sampling rate of T = 0.01 second

$$x(t) = 6 + 5\sin(4t)$$

Use superposition

x(t) = 6

s = 0

z = exp(sT) = 1

X = 6

$$Y = \left(\frac{0.01(z+1)}{(z-0.96)(z-0.9)}\right)_{z=1} \cdot (6)$$

$$Y = 30$$

x(t) = 5 sin(4t)

s = j4

z = exp(sT) = $1\angle 0.04$ = 0.9992 + j0.0400

X = 0 - j5

$$Y = \left(\frac{0.01(z+1)}{(z-0.96)(z-0.9)}\right)_{z=1\angle 0.04} \cdot (0 - j5)$$

$$Y = -15.2944 - j6.6882$$

$$y(t) = -15.2944\cos(4t) + 6.6882\sin(4t)$$

The total answer is DC + AC

$$y(t) = 30 - 15.2944\cos(4t) + 6.6882\sin(4t)$$

Problem 3) Assume G(s) is a low-pass filter with real poles:

$$G(s) = \left(\frac{100}{(s+4)(s+5)(s+6)}\right)$$

3) Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

$$s = -4 \qquad z = e^{sT} = 0.9608$$

$$s = -5 \qquad z = e^{sT} = 0.9512$$

$$s = -6 \qquad z = e^{sT} = 0.9418$$

So, G(z) is of the form

$$G(z) = \left(\frac{k}{(z-0.9608)(z-0.9512)(z-0.9418)}\right)$$

To find k, match the DC gain

$$G(s=0) = \left(\frac{100}{(s+4)(s+5)(s+6)}\right)_{s=0} = 0.8333$$

Pick 'k' so that G(z) has the same DC gain

$$G(z=1) = \left(\frac{k}{(z-0.9608)(z-0.9512)(z-0.9418)}\right)_{z=1} = 0.8333$$

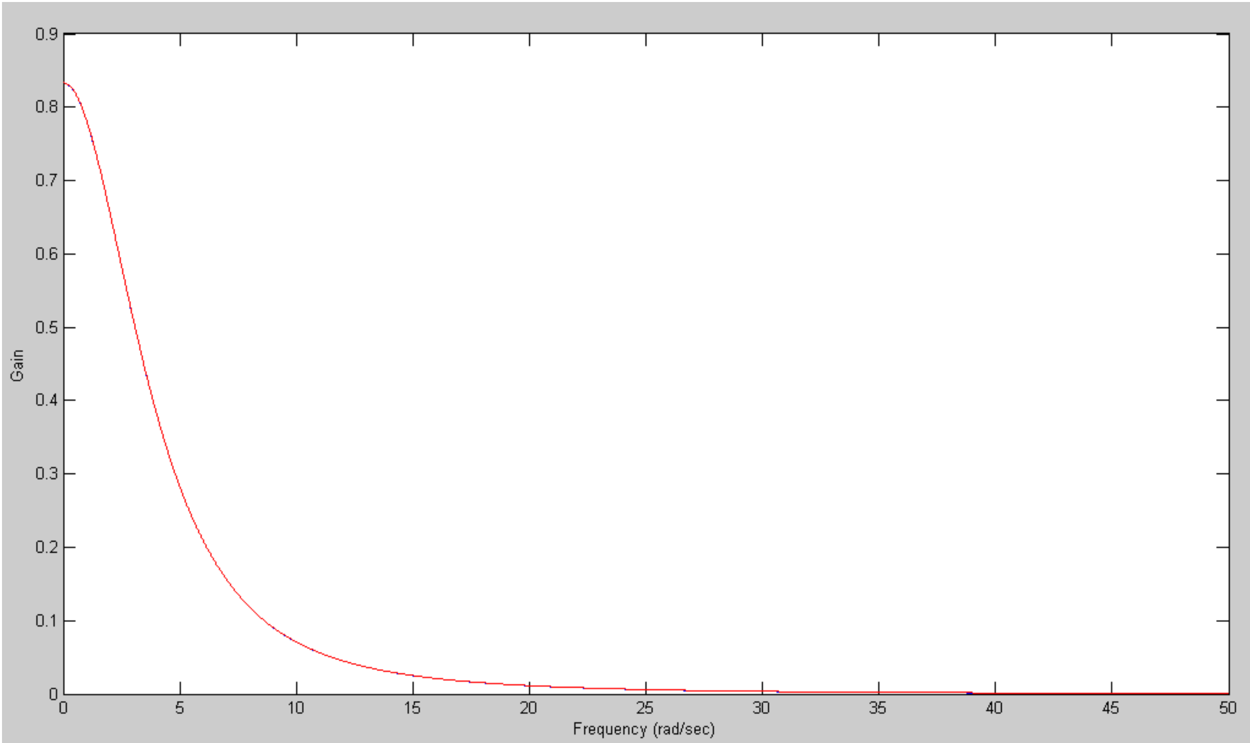
$$k = 0.0000928$$

and

$$G(z) = \left(\frac{0.0000928}{(z-0.9608)(z-0.9512)(z-0.9418)}\right)$$

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

```
>> w = [0:0.01:50]';
>> s = j*w;
>> T = 0.01;
>> z = exp(s*T);
>> Gs = 100 ./ ( (s+4).*(s+5).*(s+6) );
>> Gz = 0.0000928 ./ ( (z-0.9608).*(z-0.9512).*(z-0.9418) );
>> plot(w,abs(Gs),'b',w,abs(Gz),'r');
>> xlabel('Frequency (rad/sec)');
>> ylabel('Gain');
```



G(s) (blue) and G(z) (red)

The gain vs. frequency is the same: it's the same filter

Problem 4) Assume G(s) is the following band-pass filter:

$$G(s) = \left(\frac{30s}{(s+2+j15)(s+2-j15)}\right)$$

Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

$$s = 0 \qquad z = e^{sT} = 1$$

$$s = -2 + j15 \qquad z = e^{sT} = 0.9692 + j0.1465$$

$$s = -2 - j15 \qquad z = e^{sT} = 0.9692 - j0.1465$$

So, the form of G(z) is

$$G(z) = \left(\frac{k(z-1)}{(z-0.9692+j0.1465)(z-0.9692-j0.1465)}\right)$$

To find k, match the gain somewhere. The DC gain is zero, so pick somewhere else (like s = j15)

$$|G(s = j15)| = \left| \left(\frac{30s}{(s+2+j15)(s+2-j15)}\right)_{s=j15} \right| = 7.4834$$

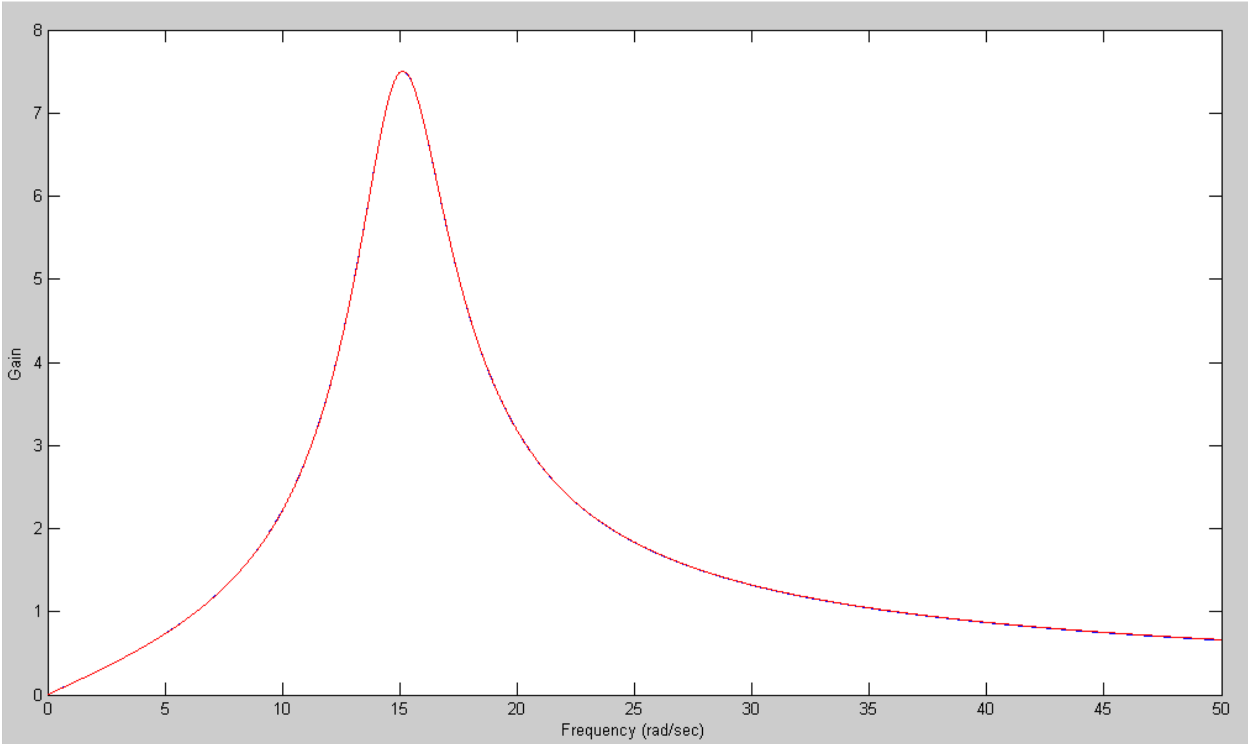
$$|G(z = e^{sT})| = \left| \left(\frac{k(z-1)}{(z-0.9692+j0.1465)(z-0.9692-j0.1465)}\right)_{z=1\angle 0.15} \right| = 7.4834$$

$$k = 0.2932$$

$$G(z) = \left(\frac{0.2932(z-1)}{(z-0.9692+j0.1465)(z-0.9692-j0.1465)}\right)$$

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

```
>> w = [0:0.01:50]';
>> s = j*w;
>> T = 0.01;
>> z = exp(s*T);
>> Gs = 30*s ./ ( (s+2+j*15).*(s+2-j*15) );
>> Gz = 0.2932*(z-1) ./ ( (z-0.9692+j*0.1465).*(z-0.9692-j*0.1465) );
>> plot(w,abs(Gs),'b',w,abs(Gz),'r');
>> xlabel('Frequency (rad/sec)');
>> ylabel('Gain');
```



G(s) (blue) & G(z) (red)

The two filters have the same gain vs. frequency: they're the same filter.

Problem 5) Write a C program to implement the digital filter, G(z)

$$Y = \left(\frac{0.2932(z-1)}{(z-0.9692+j0.1465)(z-0.9692-j0.1465)} \right) X$$

Multiply out

$$Y = \left(\frac{0.2932(z-1)}{\left(z^2-1.9384z+0.9608\right)} \right) X$$

Cross multiply

$$(z^2 - 1.9384z + 0.9608)Y = 0.2932(z-1)X$$

$$y(k+2) - 1.9384y(k+1) + 0.9608y(k) = 0.2932(x(k+1) - x(k))$$

Do a time shift (or a change of variable)

$$y(k) - 1.9384y(k-1) + 0.9608y(k-2) = 0.2932(x(k-1) - x(k-2))$$

Solve for y(k)

$$y(k) = 1.9384y(k-1) - 0.9608y(k-2) + 0.2932(x(k-1) - x(k-2))$$

That's essentially your program

```
while(1) {
   x2 = x1;
   x1 = x0;
   x0 = A2D_Read(0);

   y2 = y1;
   y1 = y0;
   y0 = 1.9384*y1 - 0.9608*y2 + 0.2932*(x1 - x2);

   D2A(y0);

   Wait_ms(10);

   }
```

Filters & Range Measurement

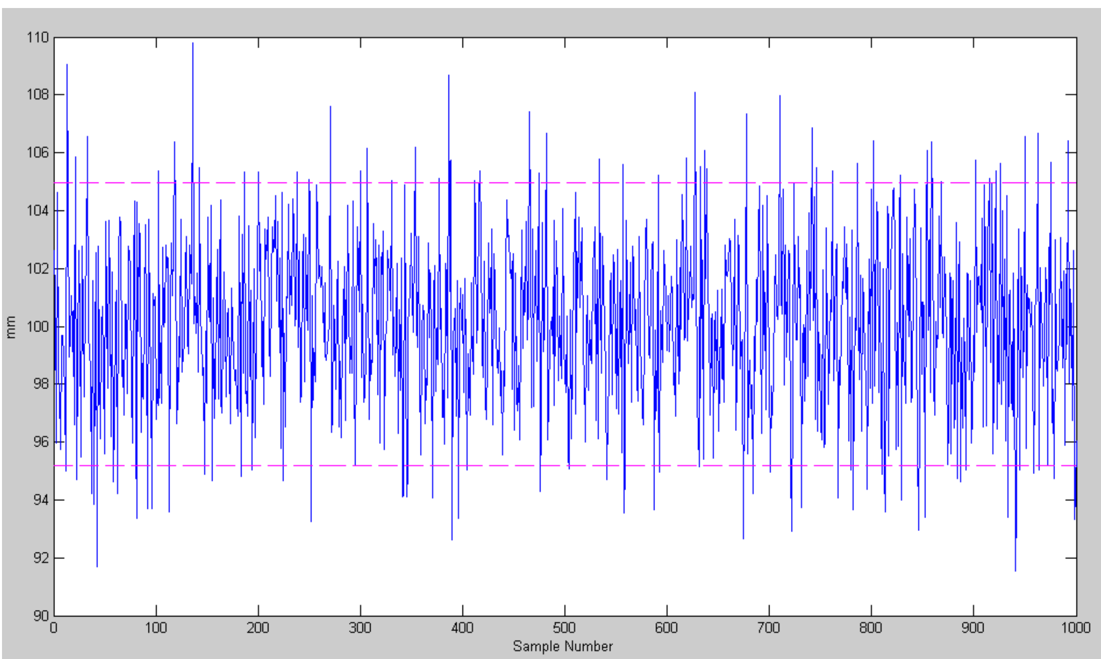
In Matlab, create data to represent ultrasonic range sensor readings at a distance of 100mm:

```
mm = 100 + 3*randn(1000,1);
```

6) For the raw data (mm), determine

- The mean
- The standard deviation
- The 90% confidence interval for the next reading.

```
>> x = 100 + 3*randn(1000,1);
>> k = [1:1000]';

>> Xx = mean(x);
>> Xx = mean(x)

Xx =  100.0820

>> Sx = std(x)

Sx =    2.9861

>> X2 = Xx + 1.64*Sx

X2 =  104.9791

>> X0 = Xx - 1.64*Sx

X0 =   95.1848

>> plot(k,x,'b',k,0*k+X2,'m--',k,0*k+X0,'m--')
>> xlabel('Sample Number');
>> ylabel('mm');
```



Raw Data (x) Along with the 90% Confidence Interval

7) Filter the data with a FIR filter (the average of the last five data points)

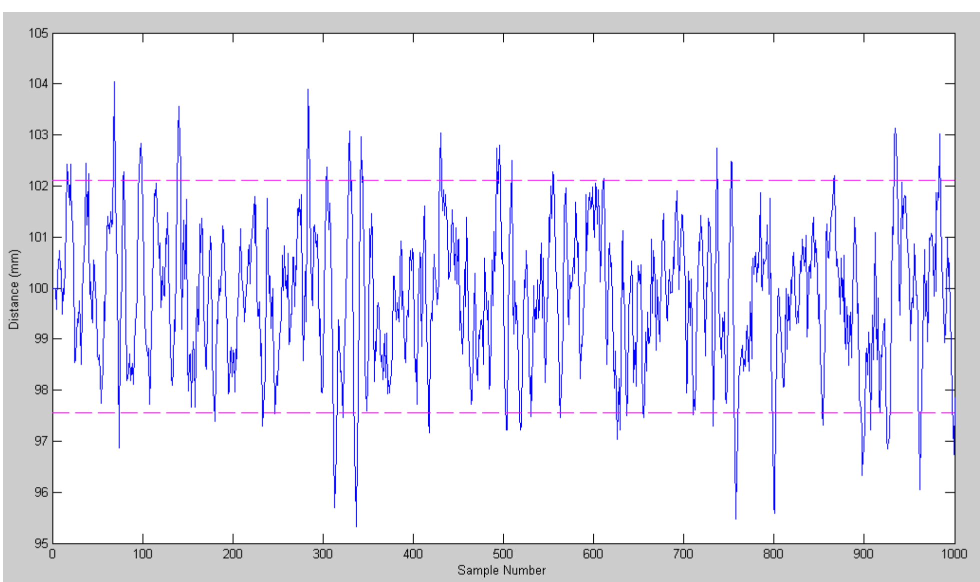
$$Y = \left(\frac{1}{5}\right)\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}\right)X$$

For the filtered data (y), determine

- The mean of y
- The standard deviation of y
- The 90% confidence interval for the next value of y

Also plot the filtered data, y(k)

```
>> x = 100 + 3*randn(1000,1);
>> y = 0*x;
>> y(1:4) = 100;
>> for k=5:1000
      y(k) = mean(x(k-4:k));
      end
>> k = [1:1000]';

>> Xy = mean(y)

Xy =   99.8276

>> Sy = std(y)

Sy =    1.3865

>> Y0 = Xy - 1.64*Sy

Y0 =   97.5538

>> Y2 = Xy + 1.64*Sy

Y2 =  102.1015

>> plot(k,y,'b',k,0*k+Y2,'m--',k,0*k+Y0,'m--')
>> xlabel('Sample Number');
>> ylabel('Distance (mm)')
```



FIR Filterd Data & 90% Confidence Interval

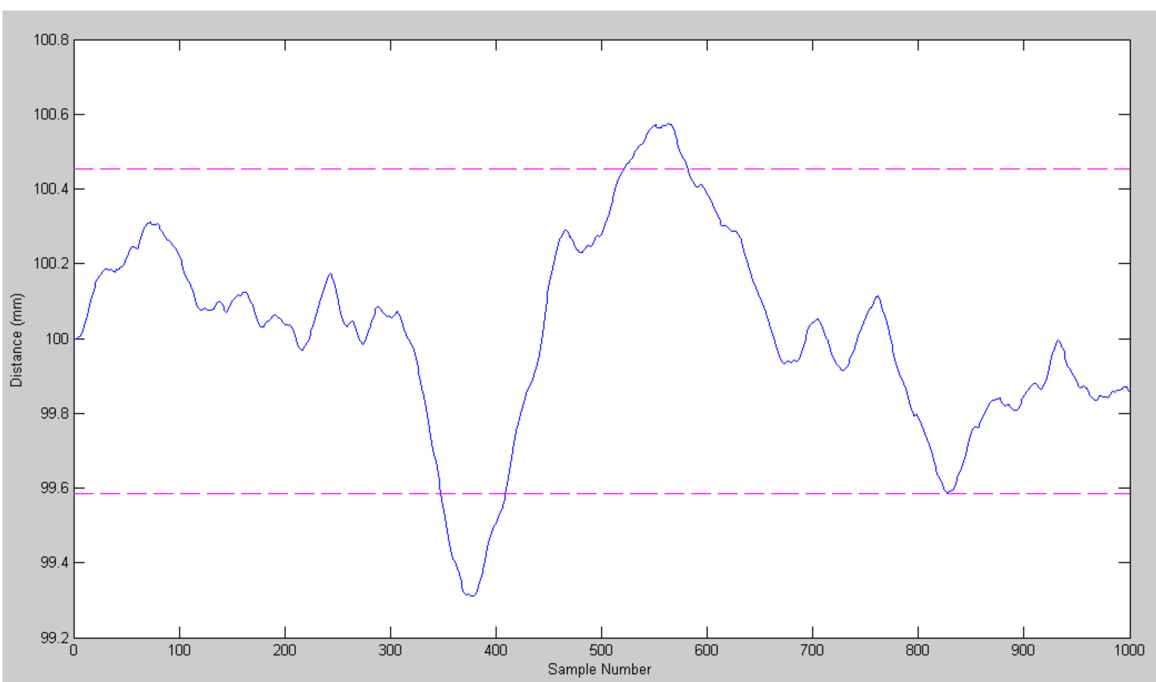
8) Filter the data with the following low-pass filter:

$$Y = \left(\frac{8}{s^2+4s+8}\right)X \qquad \text{s-plane, poles at s = -2 +/- j2}$$

$$Y = \left(\frac{0.0008}{z^2-1.9600z+0.9608}\right)X \qquad \text{same filter in the z-plane with T = 10ms}$$

In Matlab:

```
>> x = 100 + 3*randn(1000,1);
>> y = 0*x;
>> y(1:2) = 100;
>> for k=3:1000
    y(k) = 1.9600*y(k-1)-0.9608*y(k-2)+0.0008*x(k-2);
    end
>> k = [1:1000]';

>> Xy = mean(y)
Xy =  100.0203

>> Sy = std(y)
Sy =    0.2647

>> Y0 = Xy - 1.64*Sy
Y0 =   99.5862

>> Y2 = Xy + 1.64*Sy
y2 =  100.4544

>> plot(k,y,'b',k,0*k+Y2,'m--',k,0*k+Y0,'m--')
>> xlabel('Sample Number');
>> ylabel('Distance (mm)')
```



Filtered Output with an IIR Filter & 90% Confidence Interval

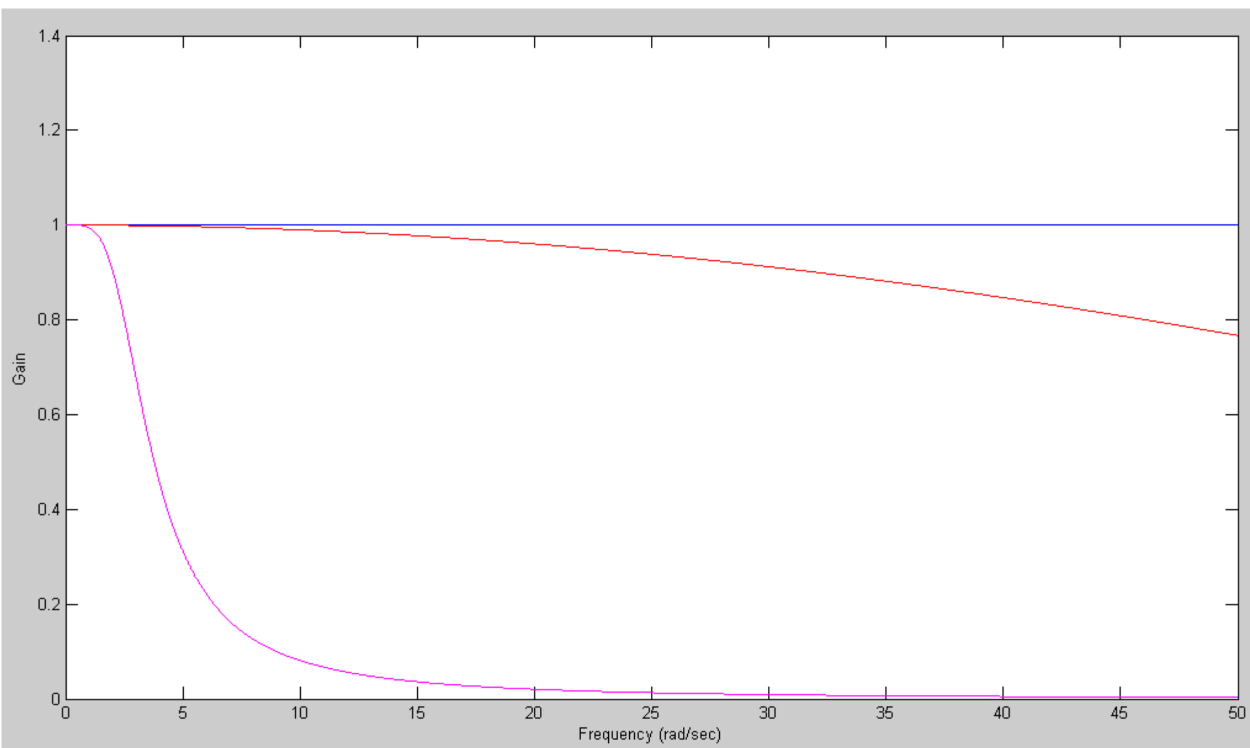
Summary:

- Taking the average of five data points is a filter. It does reduce the noise and the standard deviation.
- Running the data through a 2nd-order IIR filter works much better, however.
- You could reduce the noise even further using higher-order IIR filters

If you're going to use just a few terms, an IIR filter works much better than taking the averag

- Analysis and design is a little bit harder though

Filter	Mean	Std	90% Confidence Interval
None	100.08	2.99	95.18 .. 104.98
FIR	99.83	1.39	97.55 .. 102.10
IIR	100.02	0.26	99.59 .. 100.45



Filter Gain: No Filter (blue), FIR Filter (red), IIR Filter (pink)