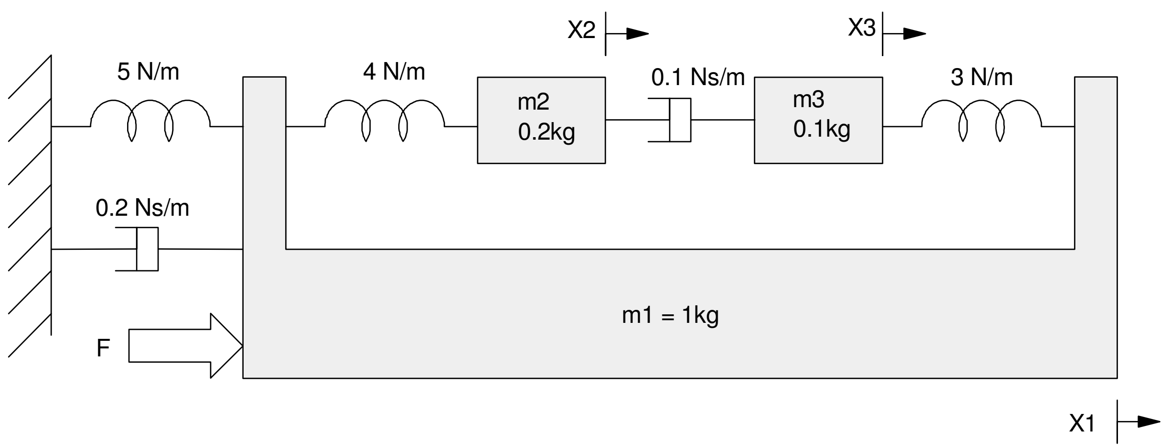
Homework #6: ECE 461/661

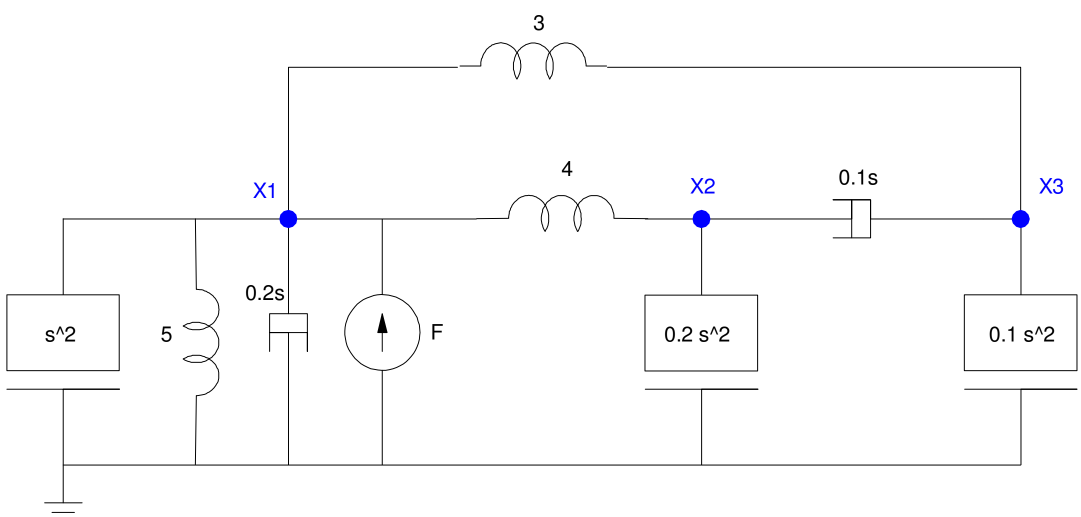
Mass-Spring Systems, Rotational Systems, DC Motors.   Due Monday, September 26th

Mass Spring systems

1) (20pt) For the following mass-spring system:



Draw the circuit equivalent



Write the dynamics (the voltage node equations)

$$(s^2+5+0.2s+3+4)X_1-(4)X_2-(3)X_3=F$$

$$(0.2s^2+0.1s+4)X_2-(4)X_1-(0.1s)X_3=0$$

$$(0.1s^2+0.1s+3)X_3-(3)X_1-(0.1s)X_2=0$$

Solve for the highest derivative

$$s^2X_1=-(0.2s+12)X_1+(4)X_2+(3)X_3+F$$

$$s^2X_2=-(0.5s+20)X_2+(20)X_1+(0.5s)X_3$$

$$s^2X_3=-(s+30)X_3+(30)X_1+(s)X_2=0$$

Express in matrix form

$$\begin{bmatrix} sX_1 \\ sX_2 \\ sX_3 \\ \cdots \\ s^2X_1 \\ s^2X_2 \\ s^2X_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ -12 & 4 & 3 & \vdots & -0.2 & 0 & 0 \\ 20 & -20 & 0 & \vdots & 0 & -0.5 & 0.5 \\ 30 & 0 & -30 & \vdots & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 1 \\ 0 \\ 0 \end{bmatrix} F$$

Find the transfer function from F to X2

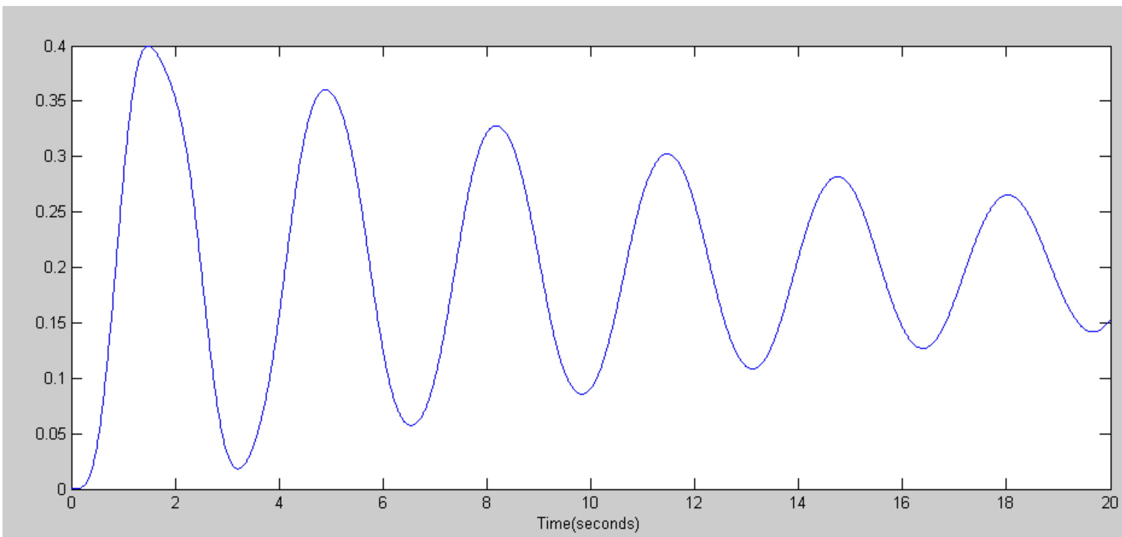
```
>> Z = zeros(3,3);
>> I = eye(3,3);
>> K = [-7,4,3 ; 20,-20,0 ; 30,0,-30];
>> B = [-5.2,0,0 ; 0,-0.5,0.5 ; 0,1,-1];
>> K = [-12,4,3 ; 20,-20,0 ; 30,0,-30];
>> B = [-0.2,0,0 ; 0,-0.5,0.5 ; 0,1,-1];
>> A = [Z,I ; K,B]

         0          0          0     1.0000          0          0
         0          0          0          0     1.0000          0
         0          0          0          0          0     1.0000
  -12.0000     4.0000     3.0000    -0.2000          0          0
   20.0000   -20.0000          0          0    -0.5000     0.5000
   30.0000          0   -30.0000          0     1.0000    -1.0000

>> eig(A)

  -0.1934 + 5.8193i
  -0.1934 - 5.8193i
  -0.0702 + 1.9171i
  -0.0702 - 1.9171i
  -0.5864 + 4.8685i
  -0.5864 - 4.8685i
```

```
>> B = [0;0;0;1;0;0]
>> C = [0,1,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)
```

```
                       20 (s^2 + 1.75s + 30)
--------------------------------------------------------------------
(s^2 + 0.1404s + 3.68) (s^2 + 1.173s + 24.05) (s^2 + 0.3867s + 33.9)

>>
```

Plot the step response from F to X2

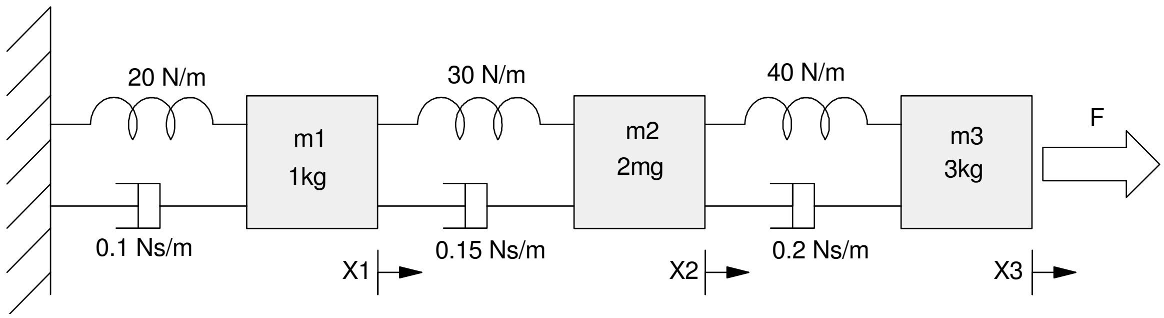
```
>> t = [0:0.01:20]';
>> y = step(G,t);
>> plot(t,y);
>> xlabel('Time(seconds)');
```



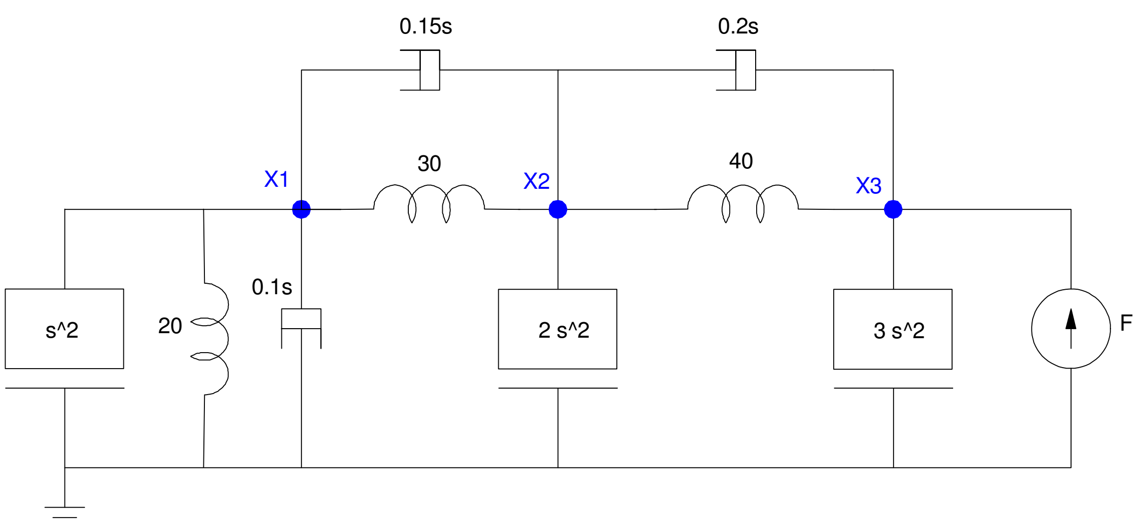
2) (20pt) Draw the circuit equivalent for the following mass-spring systems.

- Express the dynamics in state-space form
- Find the transfer function from F to X3

Plot the step response from F to X3



Draw the circuit equivalent



Write the node equations (write the system dynamics)

$$(s^2 + 20 + 0.1s + 30 + 0.15s)X_1 - (30 + 0.15s)X_2 = 0$$

$$(2s^2 + 30 + 0.15s + 40 + 0.2s)X_2 - (30 + 0.15s)X_1 - (40 + 0.2s)X_3 = 0$$

$$(3s^2 + 40 + 0.2s)X_3 - (40 + 0.2s)X_2 = F$$

Solve for the highest derivative

$$s^2X_1 = -(0.25s+50)X_1 + (0.15s+30)X_2 = 0$$

$$s^2X_2 = -(0.175s+35)X_2 + (0.075s+15)X_1 + (0.1s+20)X_3 = 0$$

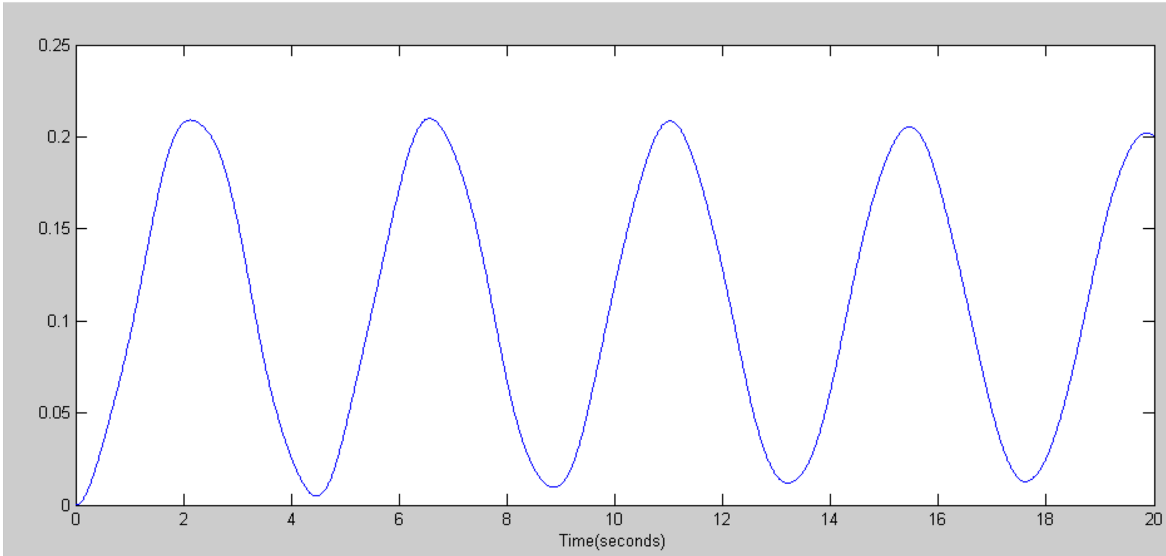
$$s^2X_3 = -(13.333+0.067s)X_3 + (13.33+0.067s)X_2 + 0.333F$$

Place in matrix form

$$\begin{bmatrix} sX_1 \\ sX_2 \\ sX_3 \\ \cdots \\ s^2X_1 \\ s^2X_2 \\ s^2X_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ -50 & 30 & 0 & \vdots & -0.25 & 0.15 & 0 \\ 15 & -35 & 20 & \vdots & 0.075 & -0.175 & 0.1 \\ 0 & 13.33 & -13.33 & \vdots & 0 & 0.067 & -0.067 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ 0.333 \end{bmatrix} F$$

Place into Matlab and find the transfer function

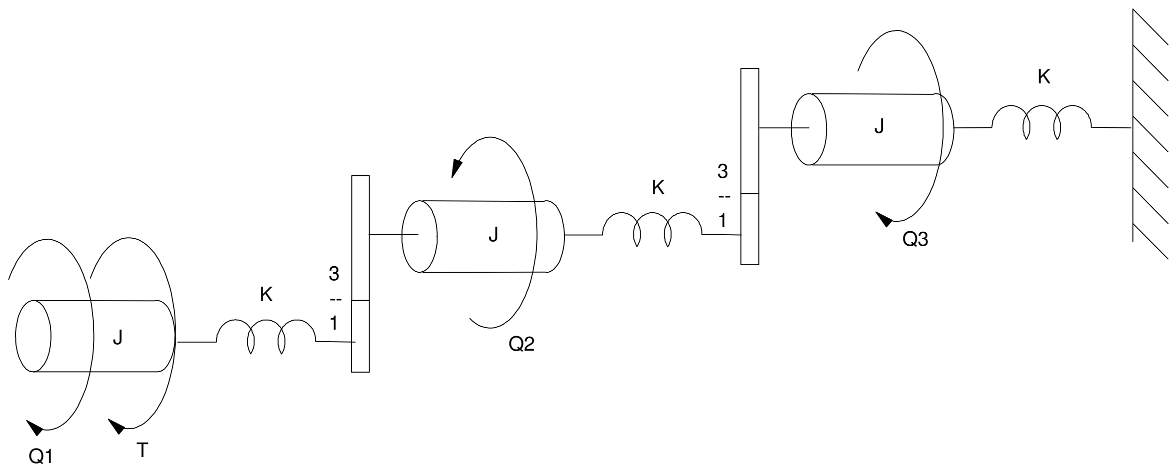
```
>> K = [-50,30,0 ; 15,-35,20 ; 0,13.33,-13.33];
>> B = [-0.25,0.15,0 ; 0.075,-0.175,0.1 ; 0,0.067,-0.067];
>> A = [Z,I ; K,B]

         0          0          0     1.0000          0          0
         0          0          0          0     1.0000          0
         0          0          0          0          0     1.0000
  -50.0000    30.0000          0    -0.2500     0.1500          0
   15.0000   -35.0000    20.0000     0.0750    -0.1750     0.1000
         0    13.3300   -13.3300          0     0.0670    -0.0670

>> eig(A)

  -0.1670 + 8.1708i
  -0.1670 - 8.1708i
  -0.0739 + 5.4320i
  -0.0739 - 5.4320i
  -0.0051 + 1.4244i
  -0.0051 - 1.4244i

>>
>> B = [0;0;0;0;0;0.333];
>> C = [0,0,1,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

              0.333 (s^2 + 0.1s + 20) (s^2 + 0.325s + 65)
----------------------------------------------------------------------
(s^2 + 0.01018s + 2.029) (s^2 + 0.1478s + 29.51) (s^2 + 0.334s + 66.79)

>> t = [0:0.01:20]';
>> y = step(G,t);
>> plot(t,y);
>> xlabel('Time(seconds)');
>>
```


0.25
0.2
0.15
0.1
0.05
0
0
2
4
6
8
10
12
14
16
18
20
Time(seconds)

Rotational Systems

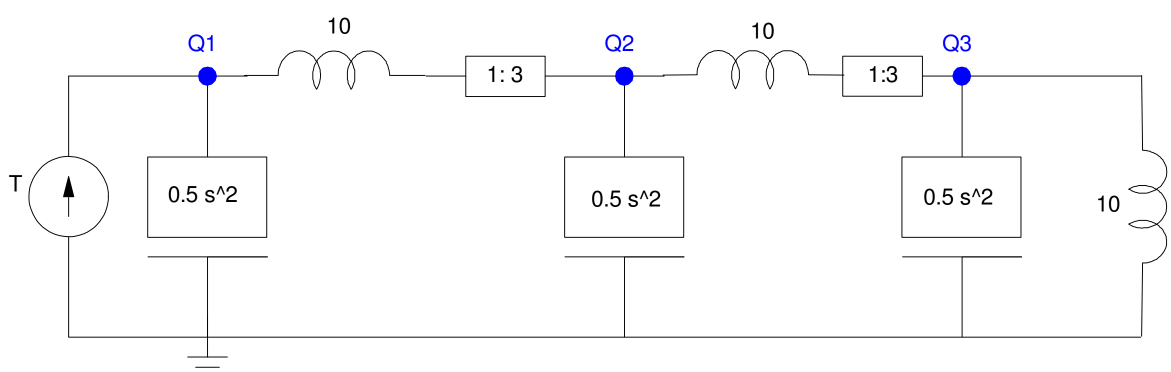
3) Draw the circuit equivalent for the following rotational system.

- Express the dynamics in state-space form
- Find the transfer function from T to Q1
- Plot the step response from T to Q1

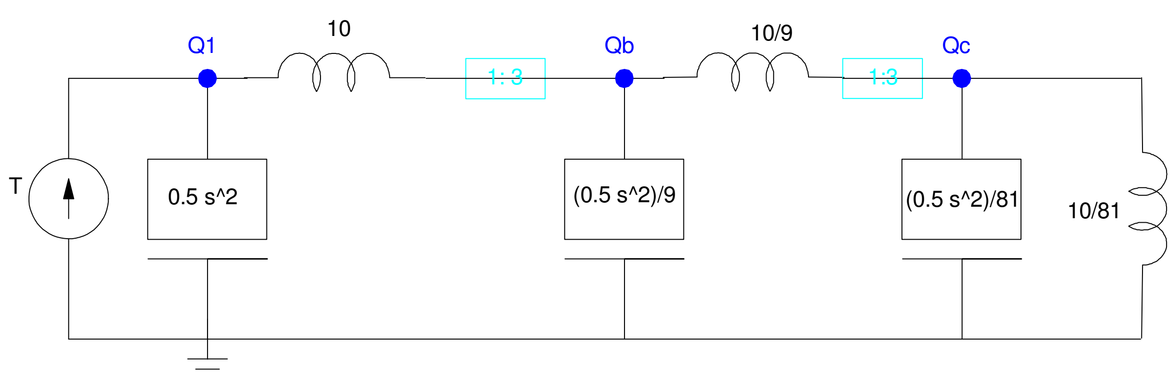


Problem 3: J = 0.5 Kg m / s^2. K = 10 Nm/rad

Draw the circuit equivalent



Remove the gears



Write the voltage node equations

$$(0.5s^2+10)\theta_1-(10)\theta_b=T$$

$$\left(\left(\frac{0.5}{9}\right)s^2+10+\frac{10}{9}\right)\theta_b-(10)\theta_1-\left(\frac{10}{9}\right)\theta_c=0$$

$$\left(\left(\frac{0.5}{81}\right)s^2+\frac{10}{9}+\frac{10}{81}\right)\theta_c-\left(\frac{10}{9}\right)\theta_b=0$$

Solve for the highest derivative

$$s^2\theta_1=-(20)\theta_1+(20)\theta_b+2T$$

$$s^2\theta_b=-(200)\theta_b+(180)\theta_1+(20)\theta_c$$

$$s^2\theta_c=-200\theta_c-180\theta_b$$

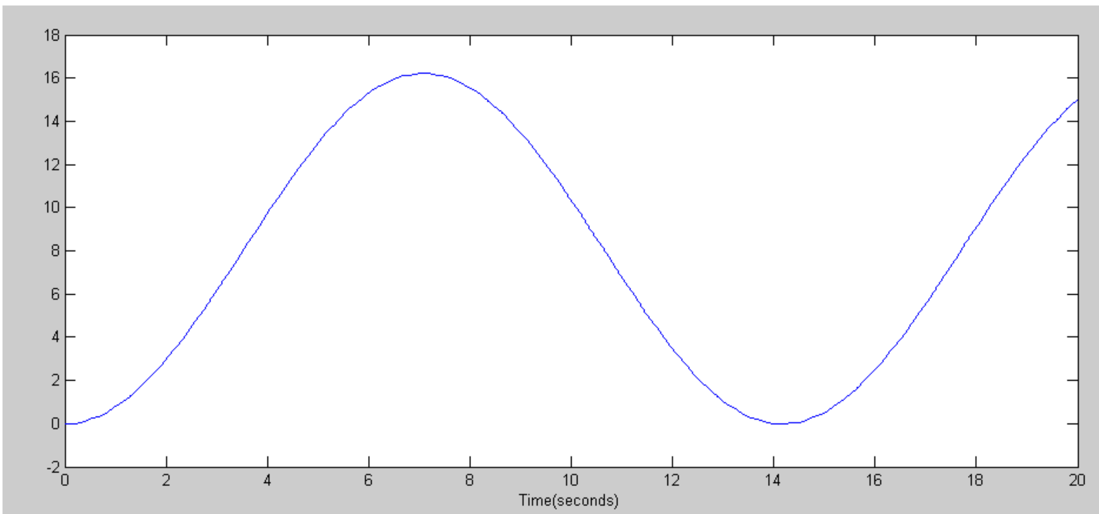
Place in matrix form

$$\begin{bmatrix} s\theta_1 \\ s\theta_b \\ s\theta_c \\ \cdots \\ s^2\theta_1 \\ s^2\theta_b \\ s^2\theta_c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ -20 & 20 & 0 & \vdots & 0 & 0 & 0 \\ 180 & -200 & 20 & \vdots & 0 & 0 & 0 \\ 0 & 180 & -200 & \vdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_b \\ \theta_c \\ \cdots \\ s\theta_1 \\ s\theta_b \\ s\theta_c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 2 \\ 0 \\ 0 \end{bmatrix} T$$

Place in Matlab and solve

```
>> Z = zeros(3,3);
>> I = eye(3,3);
>> K = [-20,20,0 ; 180,-200,20 ; 0,180,-200];
>> A = [Z,I ; K,Z]

A =

     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
   -20    20     0     0     0     0
   180  -200    20     0     0     0
     0   180  -200     0     0     0

>> eig(A)

ans =

   0.0000 +16.3617i
   0.0000 -16.3617i
  -0.0000 +12.3328i
  -0.0000 -12.3328i
   0.0000 + 0.4433i
   0.0000 - 0.4433i
```

```
>> B = [0;0;0;2;0;0];
>> C = [0,0,1,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)


                  64800
-----------------------------------------
(s^2 + 0.1965) (s^2 + 152.1) (s^2 + 267.7)

>> t = [0:0.01:20]';
>> y = step(G,t);
>> plot(t,y);
>> xlabel('Time(seconds)');
>>
```



Motors

4) Find the transfer function for the following DC servo motor

http://www.baldor.com/catalog/CDP3335

Allen Bradley CDP3335: 1/2 HP DC Servo Motor

- $1243 ea
- Armature Resistance = Ra = 0.664 Ohms
- Armature Inductance = La = 5.119mH
- Armature Inertia = J = 6.318 lb-ft2
- 4.6A @ 90V @ 2426 rpm @ 1 ft-lb load
- Weight 26.0 lb

Convert to metric

$$J = 6.318 \cdot lb \cdot ft^2 \cdot \left(\frac{0.454kg}{lb}\right)\left(\frac{0.3048m}{ft}\right)^2$$

$$J = 0.2665 \cdot kg \cdot m^2$$

$$P = 4.6A \cdot 90V = 2426rpm \cdot 1 \cdot ft \cdot lb$$

$$414W = (2624)\left(\frac{rev}{\min}\right)\left(\frac{1\min}{60s}\right)\left(\frac{2\pi rad}{rev}\right) \cdot 1 \cdot ft \cdot lb\left(\frac{0.3048m}{ft}\right)\left(\frac{4.449N}{lb}\right)$$

$$414W = \left(274.78 \cdot \frac{rad}{\sec}\right) \cdot (1.3651 \cdot Nm)$$

$$414W = 375.10W$$

38.89W is missing. The extra power must be due to the motor losses

$$38.898W = T \cdot \omega = (D\omega) \cdot \omega$$

$$38.898W = D \cdot \left(274.47\frac{rad}{\sec}\right)^2$$

$$D = 0.00051517\frac{Nm}{rad/\sec}$$

Torque Constant

$$K_t\omega = V_a - I_aR_a$$

$$K_t \cdot 274.78\frac{rad}{\sec} = 90V - 4.6A \cdot 0.664\Omega$$

$$K_t = 0.3164\frac{V}{rad/\sec} = 0.3164\frac{Nm}{A}$$

Another way to get here (ignores the friction losses (D) so it underestmates Kt)

$$T = K_tI_a$$

$$1 \cdot ft \cdot lb\left(\frac{0.3048m}{ft}\right)\left(\frac{4.449N}{lb}\right) = K_t \cdot 4.6A$$

$$K_t = 0.2947\frac{Nm}{A}$$

(slightly low due to ignoring the losses due to D)

Plugging in the numbers

$$\omega = \left(\frac{K_t}{(Js+D)(Ls+R)+K_t^2}\right)V_a$$

$$\omega = \left(\frac{K_t}{(JL)s^2+(JR+LD)s+\left(DR+K_t^2\right)}\right)V_a$$

```
>> L = 0.005119;
>> R = 0.664;
>> J = 0.2665;
>> D = 0.00051517;
>> Kt = 0.3164;
>> G = tf(Kt, [J*L, (J*R+L*D), D*R+Kt^2])

            0.3164
-------------------------------
0.001364 s^2 + 0.177 s + 0.1005

>> zpk(G)

      231.9285
--------------------
(s+129.1) (s+0.5702)

>>
```

5) Assume this motor is used to power an electric bicycle at 20mph

- Motor speed @ 20mph = 2426 rpm
- Gear (wheel) used to convert 2426 rpm to 20mph
- Bicycle weight = 100kg

What is the gear reduction (wheel diameter) to convert 2426rpm to 20mph?

$$v = r\omega$$

$$20\left(\frac{mile}{hr}\right)\left(\frac{1609.3m}{mile}\right)\left(\frac{1h}{3600s}\right) = 8.9406\frac{m}{s}$$

$$2426\left(\frac{rev}{\min}\right)\left(\frac{2\pi rad}{rev}\right)\left(\frac{1\min}{60\sec}\right) = 254.05\left(\frac{rad}{\sec}\right)$$

$$8.9406\frac{m}{s} = r \cdot 254.05\frac{rad}{\sec}$$

$$r = 0.03519m = 3.519cm$$

What is the inertia relative to the DC servo motor (bring the 100kg mass back to the motor through a gear)

$$1 \cdot rad = 0.03619 \cdot m$$

$$J = (0.03619)^2 100kg$$

$$J = 0.1309716 \cdot kg \cdot m^2$$

The inertia increses by 0.13097 to account for the mass of the rider

```
>> L = 0.005119;
>> R = 0.664;
>> J = 0.2665 + 0.1309716;
>> D = 0.00051517;
>> Kt = 0.3164;
>> G = tf(Kt, [J*L, (J*R+L*D), D*R+Kt^2]);
>> zpk(G)
```

What is the transfer function (dynamics) for the bicycle / servo motor combination?

```
      155.5053
--------------------
(s+129.3) (s+0.3817)
```