Homework #4: ECE 461 / 661

LaPlace Transforms.

LaPlace Transforms

1) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{10(s+3)}{(s+2)(s+5)(s+10)} \right) X$$

a) What is the differential equation relating X and Y?

Multiply out and cross multiply

$$Y = \left(\frac{10s+30}{s^3+17s^2+80s+100} \right) X$$

$$(s^3 + 17s^2 + 80s + 100)Y = (10s + 30)X$$

sY means *the derivative of Y*

$$\frac{d^3y}{dt^3} + 17\frac{d^2y}{dt^2} + 80\frac{dy}{dt} + 100y = 10\frac{dx}{dt} + 30x$$

or

$$y''' + 17y'' + 80y' + 100y = 10x' + 30x$$

b) Determine y(t) assuming

$$x(t) = 4\cos(3t) + 5\sin(3t)$$

This is a phasor problem

$$Y = \left(\frac{10(s+3)}{(s+2)(s+5)(s+10)} \right) X$$

is true for all 's'. In this case

$$s = j3$$
$$X = 4 - j5$$

(real = cosine, imag = -sine)

$$Y = \left(\frac{10(s+3)}{(s+2)(s+5)(s+10)} \right)_{s=j3} \cdot (4 - j5)$$

$$Y = -0.430 - j1.161$$

meaning

$$y(t) = -0.430\cos(3t) + 1.161\sin(3t)$$

c) Determine y(t) assuming x(t) is a unit step input

$$x(t) = u(t)$$

This is a LaPlace problem. The LaPlace transfor for x(t) is

$$X(s) = \left(\frac{1}{s} \right)$$

$$Y = \left(\frac{10(s+3)}{(s+2)(s+5)(s+10)} \right) \left(\frac{1}{s} \right)$$

Use partial fractions

$$Y = \left(\frac{A}{s} \right) + \left(\frac{B}{s+2} \right) + \left(\frac{C}{s+5} \right) + \left(\frac{D}{s+10} \right)$$

$$Y = \left(\frac{0.30}{s} \right) + \left(\frac{-0.208}{s+2} \right) + \left(\frac{-0.267}{s+5} \right) + \left(\frac{0.175}{s+10} \right)$$

Convert back to time

$$y(t) = (0.3 - 0.208e^{-2t} - 0.267e^{-5t} + 0.175e^{-10t})u(t)$$

2) Assume X and Y are related by the following transfer function:

$$Y = \left(\frac{100}{(s+1+j5)(s+1-j5)(s+30)} \right) X$$

a) Use 2nd-order approximations to determine
   - The 2% settling time
   - The percent overshoot for a step input
   - The steady-state output for a step input ($x(t) = u(t)$)

The dominant pole is

$$s = -1 \pm j5$$

The 2% settling time is

$$T_s = \frac{4}{real(s)} = 4 \sec$$

The damping ratio comes from the angle of the dominant pole

$$\theta = \arctan\left(\frac{imag}{real} \right) = \arctan\left(\frac{5}{1} \right) = 78.69^0$$

$$\zeta = \cos(78.69^0) = 0.196$$

The overshoot is 53.3%

$$OS = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \right) = 0.533$$

The steady-state output is the DC gain

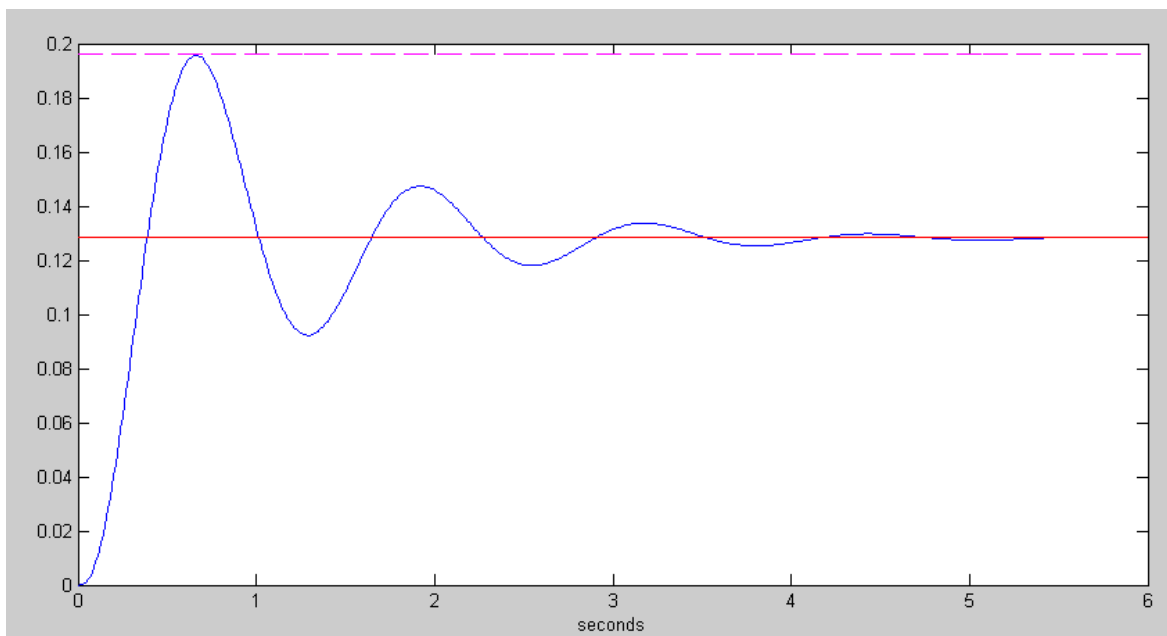$$G(s = 0) = \left(\frac{100}{(s+1+j5)(s+1-j5)(s+30)} \right)_{s=0} = 0.128$$

b) Check your answers using the 3rd order model and Matlab, Simulink, of VisSim (your pick)

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>> G = zpk([],[-1+j*5,-1-j*5,-30],100)

Zero/pole/gain:
          100
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(s+30) (s^2 + 2s + 26)

>> t = [0:0.01:6]';
>> y = step(G,t);
>> plot(t,y);
>> xlabel('seconds');

>> DC = evalfr(G,0);
>> plot(t,y,'b',t,DC+t*0,'r',t,DC*1.53+t*0,'m--')
>> xlabel('seconds');
```

3) Determine the transfer function for a system with the following step response:

This is a 1st-order system (no oscillations) meaning

$$Y = \left(\frac{a}{s+b}\right)X$$

There are 2 unknowns. We need to take two measurements to find a and b

The DC gain is 2.95

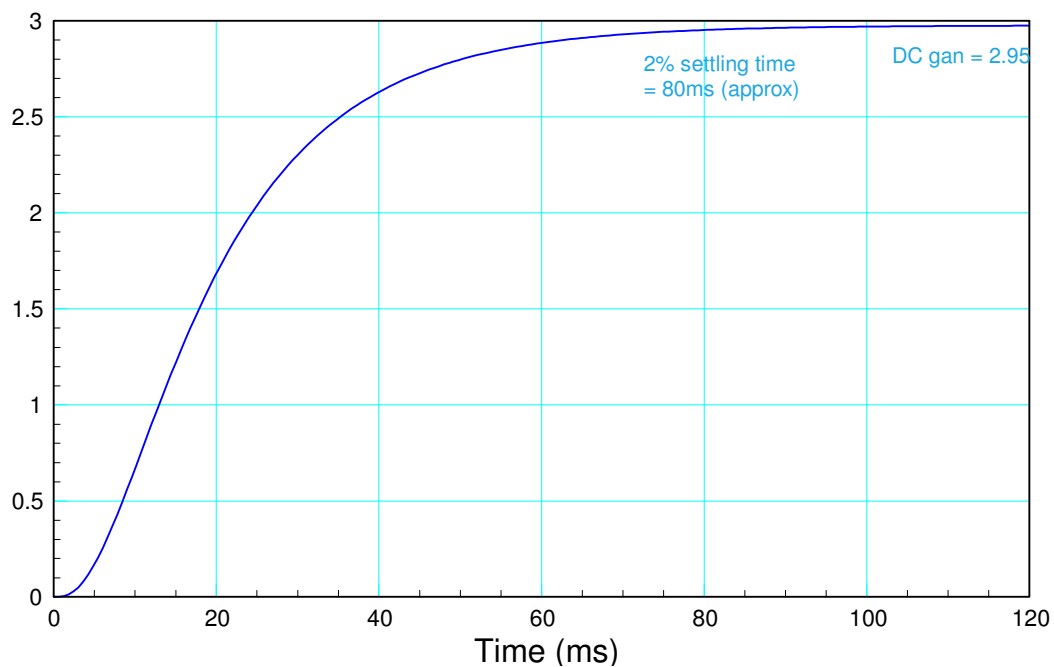$$\left(\frac{a}{s+b}\right)_{s=0} = \left(\frac{a}{b}\right) = 2.95$$

The 2% settling time is about 80ms

$$T_s = 80ms = \frac{4}{b}$$

$$b = 50$$

which makes a = 147.5 (*sets the DC gain to 2.95*)

$$Y \approx \left(\frac{147.5}{s+50}\right)$$

4) Determine the transfer function for a system with the following step response:

This is a 2nd-order system (it oscillates, meaning the poles are complex). In general

$$Y = \left(\frac{k}{(s+a+jb)(s+a-jb)} \right) X$$

There are 3 unknowns - we need to take 3 measurements

DC Gain = 3.4

$$\left(\frac{k}{(s+a+jb)(s+a-jb)} \right)_{s=0} = \left(\frac{k}{a^2+b^2} \right) = 3.4$$

Frequency of oscillation (imag(pole))

$$b = \left(\frac{3 \text{ cycles}}{82\text{ms}} \right) \cdot 2\pi = 229.9 \frac{rad}{\text{sec}}$$

2% settlimg time = 110ms (approx)

$$T_s = \frac{4}{a} = 110ms$$

$$a = 36.4$$

so

$$G(s) \approx \left(\frac{182,648}{(s+36.4+j228.9)(s+36.4-j228.9)} \right)$$

the numerator sets the DC gain to 3.4