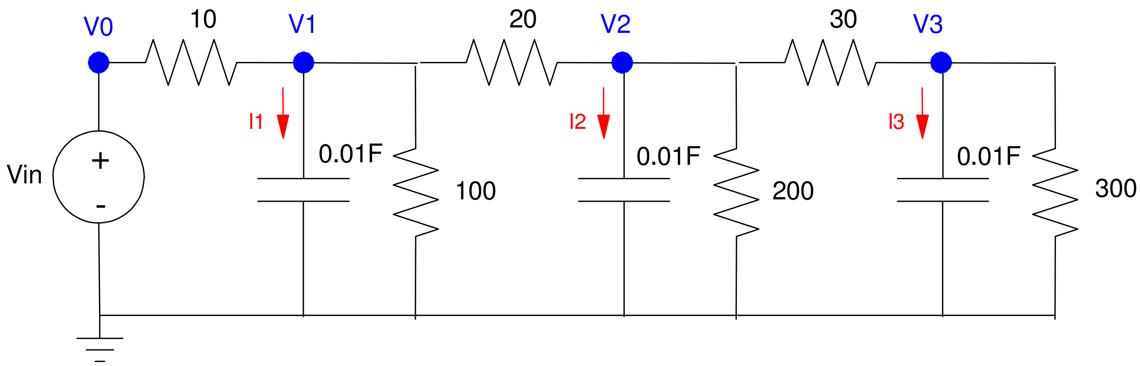
Homework #6: ECE 461/661

Mass-Spring Systems, Rotational Systems

Electrical Ciruits

1) Using state-space methods, find the transfer function from Vin to V3



$$I_1 = CsV_1 = \left(\frac{V_0 - V_1}{10}\right) + \left(\frac{V_2 - V_1}{20}\right) - \left(\frac{V_1}{100}\right)$$

$$I_2 = CsV_2 = \left(\frac{V_1 - V_2}{20}\right) + \left(\frac{V_3 - V_2}{30}\right) - \left(\frac{V_2}{200}\right)$$

$$I_3 = CsV_3 = \left(\frac{V_2 - V_3}{30}\right) - \left(\frac{V_3}{300}\right)$$

Simplifying and solving for sVx

$$sV_1 = 10V_0 - 16V_1 + 5V_2$$

$$sV_2 = 5V_1 - 8.833V_2 + 3.333V_3$$

$$sV_3 = 3.333V_2 - 3.667V_3$$

In state-space

$$\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \end{bmatrix} = \begin{bmatrix} -16 & 5 & 0 \\ 5 & -8.833 & 3.333 \\ 0 & 3.333 & -3.667 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} V_0$$

$$y = V_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Use Matlab to find the transfer funciton

```
>> A = [-16,5,0 ; 5,-8.833,3.333 ; 0,3.333,-3.667]

A =

  -16.0000    5.0000         0
    5.0000   -8.8330    3.3330
         0    3.3330   -3.6670

>> B = [10;0;0]

B =

    10
     0
     0

>> C = [0,0,1]

C =

     0     0     1

>> G = ss(A,B,C,0);
>> zpk(G)

           166.65
-----------------------------
(s+18.73) (s+8.138) (s+1.633)
```

5) Using state-space methods, find the transfer function from Vin to V2

All that changes is the C matrix

```
>> C = [0,1,0];
>> G = ss(A,B,C,0);
>> zpk(G)

        50 (s+3.667)
-----------------------------
(s+18.73) (s+8.138) (s+1.633)
```

Note

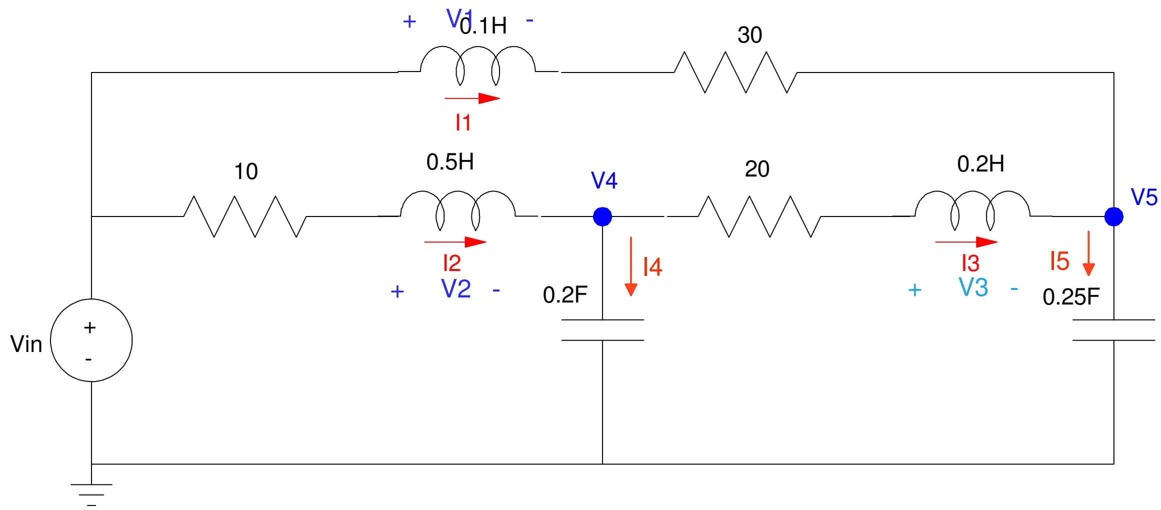
- The poles don't change when you change the output
- The zeros do change

Poles tell you how the energy in the system dissipates. Changing the output doesn't change this.

Zeros tell you how the energy states contribute to the output. Changing the output does change the zeros.

2) Express the dynamics for the following RLC circuit in state-space form.

- Find the transfr function from Vin to V5



The energy in the system is defined by the current in the inductors and the voltage across the capacitors. Let the states be { I1, I2, I3, V4, V5 }

$$V_1 = 0.1sI_1 = V_{in} - 30I_1 - V_5$$

$$V_2 = 0.5sI_2 = V_{in} - 10I_2 - V_4$$

$$V_3 = 0.2sI_3 = V_4 - 20I_3 - V_5$$

$$I_4 = 0.2sV_4 = I_2 - I_3$$

$$I_5 = 0.25sV_5 = I_1 + I_3$$

Simplifying

$$sI_1 = 10V_{in} - 300I_1 - 10V_5$$

$$sI_2 = 2V_{in} - 20I_2 - 2V_4$$

$$sI_3 = 5V_4 - 100I_3 - 5V_5$$

$$sV_4 = 5I_2 - 5I_3$$

$$sV_4 = 5I_2 - 5I_3$$

Place in matrix (state-space) form

$$\begin{bmatrix} sI_1 \\ sI_2 \\ sI_3 \\ sV_4 \\ sV_5 \end{bmatrix} = \begin{bmatrix} -300 & 0 & 0 & 0 & -10 \\ 0 & -20 & 0 & -2 & 0 \\ 0 & 0 & -100 & 5 & -5 \\ 0 & 5 & -5 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_4 \\ V_5 \end{bmatrix} + \begin{bmatrix} 10 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

Use Matlab to find the transfer funciton to V5

```
>> A = [-300,0,0,0,-10 ; 0,-20,0,-2,0 ; 0,0,-100,5,-5;0,5,-5,0,0;4,0,4,0,0]

  -300      0      0      0    -10
     0    -20      0     -2      0
     0      0   -100      5     -5
     0      5     -5      0      0
     4      0      4      0      0

>> B = [10;2;0;0;0]

    10
     2
     0
     0
     0

>> C = [0,0,0,0,1];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

          40 (s+99.88) (s+18.5) (s+1.624)
--------------------------------------------------
(s+299.9) (s+99.55) (s+19.48) (s+0.8701) (s+0.2372)
```

3) Assime Vin = 0. Specify the initial conditions so that the total energy at t = 0 is 1.0 Joules and

- The transients decay as slow as possible
- The transients decay as fast as possible

This is an eigenvalue / eigenvector problem

```
>> [M,V] = eig(A)
```

M (eigenvector)

0.9999	0.0020	**-0.0300**	0.0117	0.0001
-0.0000	0.0013	**-0.0439**	-0.0971	-0.9675
-0.0003	0.9979	**-0.0233**	0.0646	0.0159
-0.0000	0.0501	**0.4338**	0.9291	0.2524
-0.0133	-0.0402	**0.8991**	-0.3506	-0.0033

V (eigenvalues)

-299.8667	0	**0**	0	0
0	-99.5479	**0**	0	0
0	0	**-0.2372**	0	0
0	0	**0**	-0.8701	0
0	0	**0**	0	-19.4782

The red eigenvector decays the fastest. Its energy is

$$E = \Sigma\left(\tfrac{1}{2}LI^2 + \tfrac{1}{2}CV^2\right) =$$

```
>> X0 = M(:,1);
>> Energy = 0.5 * [0.1, 0.5, 0.2, 0.2, 0.25] * X0.^2

Energy =      0.0500

>> X0 / Energy
```

```
I1   19.9929A
I2   -0.0000A
I3   -0.0067A
V4   -0.0001V
V5   -0.2666V
```

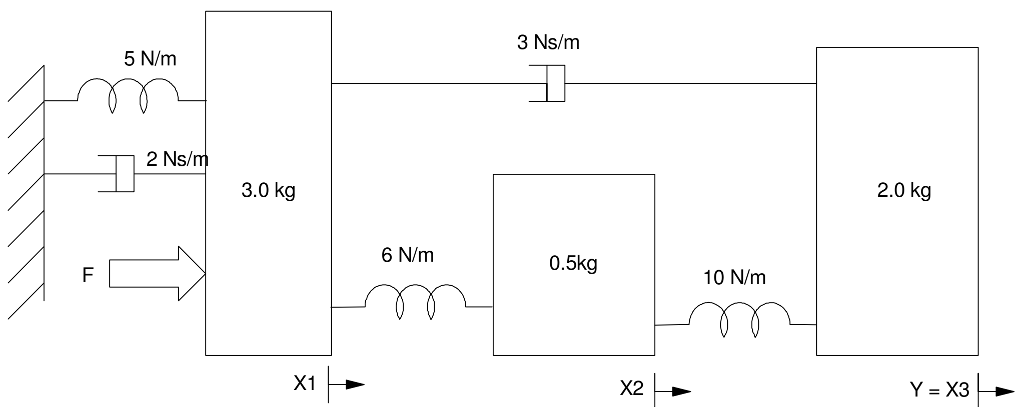
The 3rd eigenvector (blue) is the slow mode. For its initial energy to be 1.00 Joule

```
>> X0 = M(:,3);
>> Energy = 0.5 * [0.1, 0.5, 0.2, 0.2, 0.25] * X0.^2

Energy =    0.1205

>> X0 / Energy
```

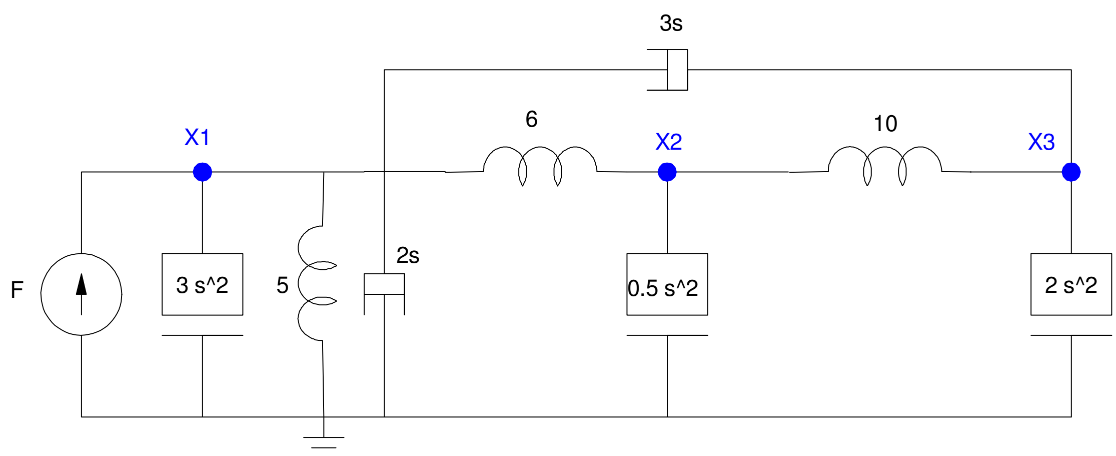
```
I1   -0.2490A
I2   -0.3645A
I3   -0.1936A
V4    3.6014V
V5    7.4645V
```

Mass Spring systems

4) For the following mass-spring system:



Draw the circuit equivalent for the following mass-spring systems.



Express the dynamics in state-space form

$$(3s^2 + 2s + 5 + 6 + 3s)X_1 - (6)X_2 - (3s)X_3 = F$$

$$(0.5s^2 + 6 + 10)X_2 - (6)X_1 - (10)X_3 = 0$$

$$(2s^2 + 3s + 10)X_3 - (3s)X_1 - (10)X_2 = 0$$

Solve for the highest derivative

$$s^2 X_1 = \left(-\tfrac{5}{3}s - \tfrac{11}{3}\right)X_1 + 2X_2 + sX_3 + \tfrac{1}{3}F$$

$$s^2 X_2 = -32X_2 + 12X_1 + 20X_3$$

$$s^2 X_3 = \left(-\tfrac{3}{2}s - 5\right)X_3 + \tfrac{3}{2}sX_1 + 5X_2$$

Place in matrix form

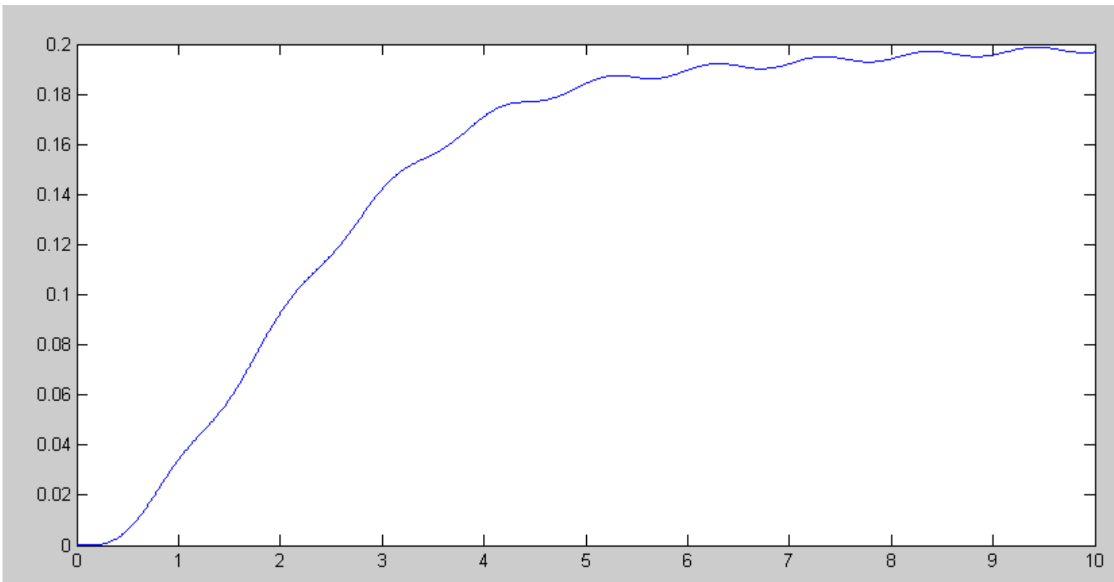
$$s\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ -\frac{11}{3} & 2 & 0 & \vdots & -\frac{5}{3} & 0 & 1 \\ 12 & -32 & 20 & \vdots & 0 & 0 & 0 \\ 0 & 5 & -5 & \vdots & \frac{3}{2} & 0 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ \frac{1}{3} \\ 0 \\ 0 \end{bmatrix} F$$

Find the transfer function from F to X2

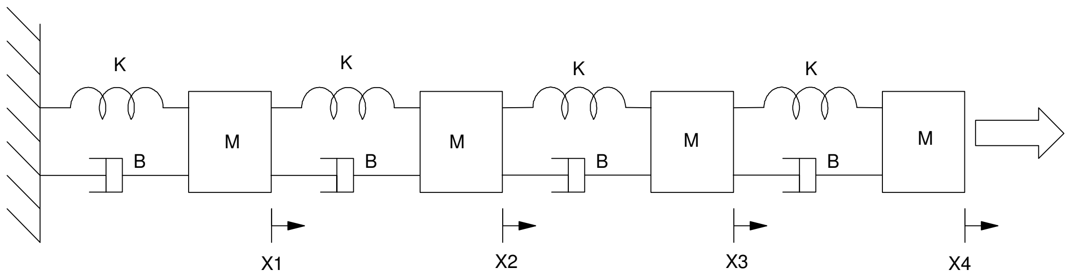
```
>> a11 = zeros(3,3);
>> a12 = eye(3,3);
>> a21 = [-11/3,2,0 ; 12,-32,20 ; 0,5,-5];
>> a22 = [-5.3,0,1 ; 0,0,0 ; 3/2,0,-3/2];
>> A = [a11,a12 ; a21,a22]

A =

         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
   -3.6667    2.0000         0   -5.3000         0    1.0000
   12.0000  -32.0000   20.0000         0         0         0
         0    5.0000   -5.0000    1.5000         0   -1.5000

>> B = [0;0;0;1/3;0;0];
>> C = [0,1,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

Zero/pole/gain:
                        4 (s^2 + 4s + 5)
-----------------------------------------------------------------
(s+4.91) (s+0.4769) (s^2 + 1.283s + 1.193) (s^2 + 0.1298s + 35.8)
```

Plot the step response from F to X2

```
>> t = [0:0.01:10]';
>> y = step(G,t);
>> plot(t,y)
>> eig(A)

  -0.0649 + 5.9832i
  -0.0649 - 5.9832i
  -4.9099
  -0.4769
  -0.6417 + 0.8838i
  -0.6417 - 0.8838i

>>
```

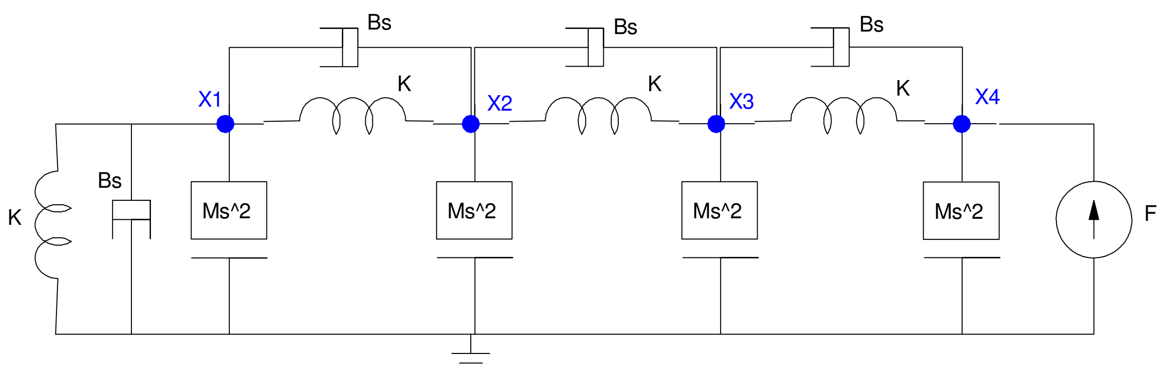


5) For the following mass-spring system...



Problem 6: M = 2.0kg, B = 0.3 Ns/m, K = 10 N/m

Draw the circuit equivalent



Express the dynamics in state-space form plug in (K/M = 5), (B/M = 0.15)

$$(Ms^2 + 2Bs + 2K)X_1 - (Bs + K)X_2 = 0$$

$$(Ms^2 + 2Bs + 2K)X_2 - (Bs + K)X_1 - (Bs + K)X_3 = 0$$

$$(Ms^2 + 2Bs + 2K)X_3 - (Bs + K)X_2 - (Bs + K)X_4 = 0$$

$$(Ms^2 + Bs + K)X_4 - (Bs + K)X_3 = F$$

Place in matix form

$$\begin{bmatrix} sX_1 \\ sX_2 \\ sX_3 \\ sX_4 \\ \cdots \\ s^2X_1 \\ s^2X_2 \\ s^2X_3 \\ s^2X_4 \end{bmatrix} = \left[\begin{array}{cccc:cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{-2K}{M} & \frac{K}{M} & 0 & 0 & \frac{-2B}{M} & \frac{B}{M} & 0 & 0 \\ \frac{K}{M} & \frac{-2K}{M} & \frac{K}{M} & 0 & \frac{B}{M} & \frac{-2B}{M} & \frac{B}{M} & 0 \\ 0 & \frac{K}{M} & \frac{-2K}{M} & \frac{K}{M} & 0 & \frac{B}{M} & \frac{-2B}{M} & \frac{B}{M} \\ 0 & 0 & \frac{K}{M} & \frac{-K}{M} & 0 & 0 & \frac{B}{M} & \frac{-B}{M} \end{array}\right] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \\ sX_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ 0 \\ \frac{1}{M} \end{bmatrix} F$$

plug into Matlab noting that

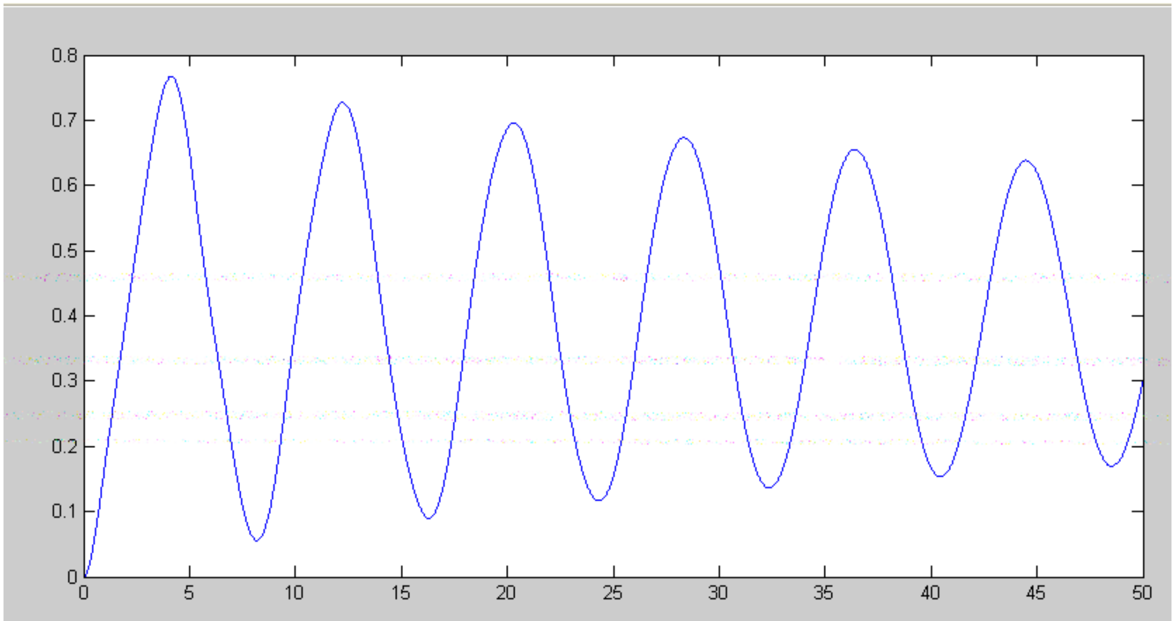
- K/M = 5
- B/M = 0.15
-

Find the transfer function from F to X4

```
>> a11 = zeros(4,4);
>> a12 = eye(4,4);
>> a21 = [-10,5,0,0 ; 5,-10,5,0 ; 0,5,-10,5 ; 0,0,5,-5];
>> a22 = [-0.3,0.15,0,0 ; 0.15,-0.3,0.15,0 ; 0,0.15,-0.3,0.15 ; 0,0,0.15,-0.15];
>> A = [a11,a12 ; a21,a22]

A =

         0         0         0         0    1.0000         0         0         0
         0         0         0         0         0    1.0000         0         0
         0         0         0         0         0         0    1.0000         0
         0         0         0         0         0         0         0    1.0000
  -10.0000    5.0000         0         0   -0.3000    0.1500         0         0
    5.0000  -10.0000    5.0000         0    0.1500   -0.3000    0.1500         0
         0    5.0000  -10.0000    5.0000         0    0.1500   -0.3000    0.1500
         0         0    5.0000   -5.0000         0         0    0.1500   -0.1500

>> B = [0;0;0;0;0;0;0;1/2]

B =

         0
         0
         0
         0
         0
         0
         0
    0.5000

>> C = [0,0,0,1,0,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
??? Error using ==> ss.ss>ss.ss at 345
The values of the "a" and "c" properties must be matrices with the same
number of columns.

>> C = [0,0,0,1,0,0,0,0];;
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)


        0.5 (s^2 + 0.08787s + 2.929) (s^2 + 0.3s + 10) (s^2 + 0.5121s + 17.07)
------------------------------------------------------------------------------------
(s^2 + 0.01809s + 0.6031) (s^2 + 0.15s + 5) (s^2 + 0.3521s + 11.74) (s^2 + 0.5298s + 17.66)

>> eig(A)


  -0.2649 + 4.1941i
  -0.2649 - 4.1941i
  -0.1760 + 3.4213i
  -0.1760 - 3.4213i
  -0.0750 + 2.2348i
  -0.0750 - 2.2348i
  -0.0090 + 0.7765i
  -0.0090 - 0.7765i


>>
```

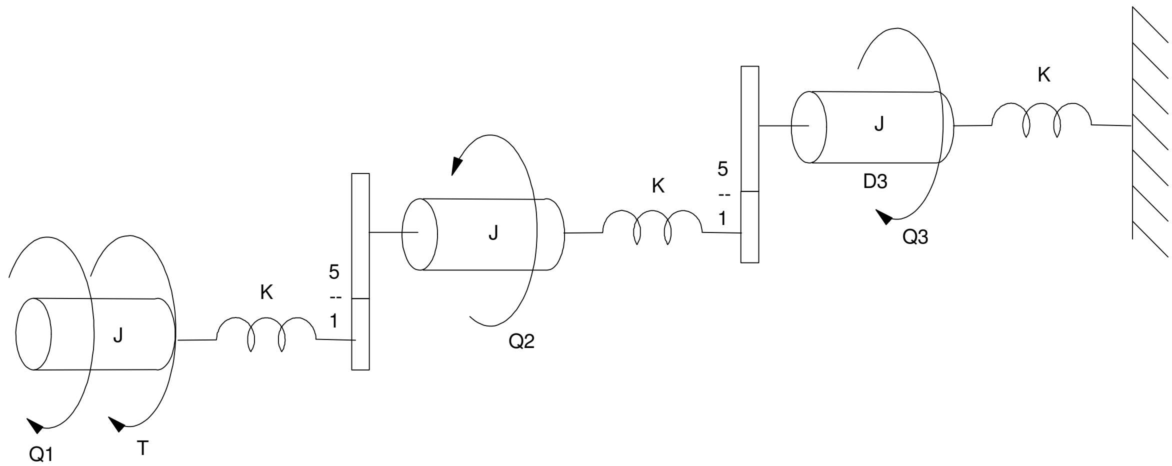
Plot the step response from F to X4

```
>> t = [0:0.01:50]';
>> y = step(G,t);
>> plot(t,y)
```



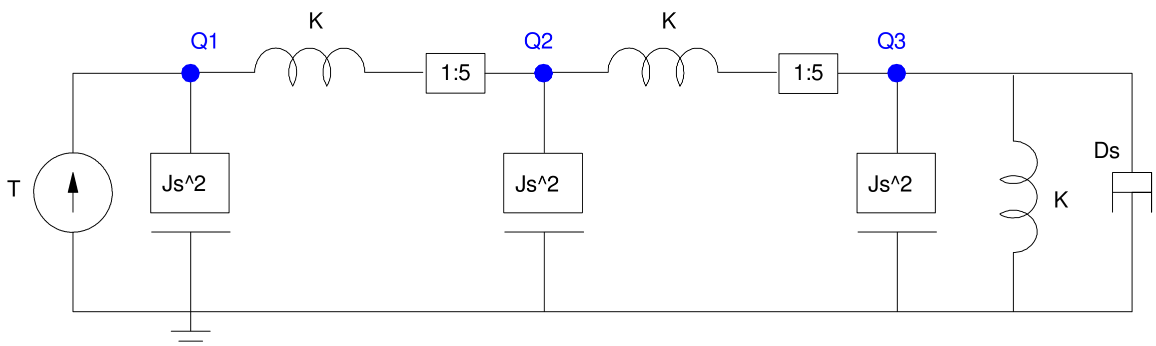
Rotational Systems

6) For the following rotational system...

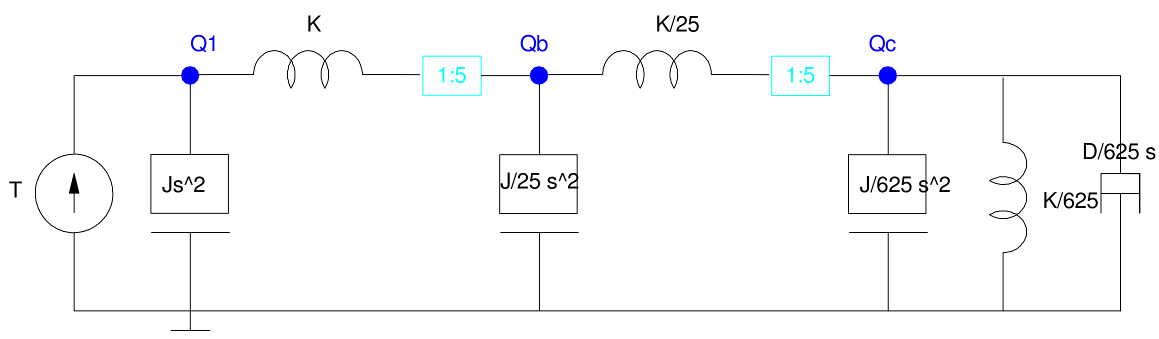


Problem 3: J = 2.5 Kg m / s^2. K = 20 Nm/rad, D3 = 1.5 Nms/ rad

Draw the circuit equivalent



Remove the gears



Write the node equations

$$(Js^2 + K)\theta_1 - (K)\theta_2 = T$$

$$\left(\frac{J}{25}s^2 + K + \frac{K}{25}\right)\theta_b - (K)\theta_1 - \left(\frac{K}{25}\right)\theta_c = 0$$

$$\left(\frac{J}{625}s^2 + \frac{D}{625}s + \frac{K}{625} + \frac{K}{25}\right)\theta_c - \left(\frac{K}{25}\right)\theta_b = 0$$

Solving for the highest derivative

Problem 3: J = 2.5 Kg m / s^2. K = 20 Nm/rad, D3 = 1.5 Nms/ rad

$$s^2\theta_1 = \left(-\frac{K}{J}\right)\theta_1 + \left(\frac{K}{J}\right)\theta_b + \left(\frac{1}{J}\right)T$$

$$s^2\theta_b = \left(\frac{25K}{J}\right)\theta_1 - \left(\frac{26K}{J}\right)\theta_b + \left(\frac{K}{J}\right)\theta_c$$

$$s^2\theta_c = \left(\frac{25K}{J}\right)\theta_b - \left(\frac{D}{J}s + \frac{26K}{J}\right)\theta_c$$

Plugging in numbers and placing in matrix form (K/J = 8, D/J = 0.6)

$$\begin{bmatrix} s\theta_1 \\ s\theta_b \\ s\theta_c \\ \cdots \\ s^2\theta_1 \\ s^2\theta_b \\ s^2\theta_c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ -8 & 8 & 0 & \vdots & 0 & 0 & 0 \\ 200 & -208 & 8 & \vdots & 0 & 0 & 0 \\ 0 & 200 & -208 & \vdots & 0 & 0 & -1.5 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_b \\ \theta_c \\ \cdots \\ s\theta_1 \\ s\theta_b \\ s\theta_c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0.4 \\ 0 \\ 0 \end{bmatrix} T$$

Find the transfer function from T to Q1

```
>> a11 = zeros(3,3);
>> a12 = eye(3,3);
>> a21 = [-8,8,0 ; 200,-208,8 ; 0,200,-208];
>> a22 = [0,0,0 ; 0,0,0 ; 0,0,-1.5];
>> A = [a11,a12 ; a21,a22]

         0          0          0     1.0000          0          0
         0          0          0          0     1.0000          0
         0          0          0          0          0     1.0000
   -8.0000     8.0000          0          0          0          0
  200.0000  -208.0000     8.0000          0          0          0
         0   200.0000  -208.0000          0          0    -1.5000

>> B = [0;0;0;0.4;0;0];
>> C = [1,0,0,0,0,0];
>> D = 0;
```

```
>> G = ss(A,B,C,D);
>> zpk(G)

           0.4 (s^2 + 0.7555s + 169.2) (s^2 + 0.7445s + 246.2)
--------------------------------------------------------------------------
(s^2 + 0.002125s + 0.0118) (s^2 + 0.8275s + 173.8) (s^2 + 0.6703s + 249.6)

>> eig(A)

  -0.3352 +15.7959i
  -0.3352 -15.7959i
  -0.4138 +13.1771i
  -0.4138 -13.1771i
  -0.0011 + 0.1086i
  -0.0011 - 0.1086i

>>
```

Plot the step response from T to Q1

```
>> t = [0:0.1:500]';
>> y = step(G,t);
>> plot(t,y)
```