ECE 463/663 - Homework #11

LQR Observers. Due Wednesday, April 20th, 2022

Kalman Filters

Cart and Pendulum (HW #4): The dynamics for a cart and pendulum system with sensor and input noise is as follows

$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -14.7 & 0 & 0 \\ 0 & 24.5 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix}(F + n_u)$$

$$y_1 = x + n_x$$

$$y_2 = \theta + n_\theta$$

where there is Gaussian noise at the input and output

$n_u \sim N(0, 0.02^2)$ *mean zero, standard deviation 0.02*

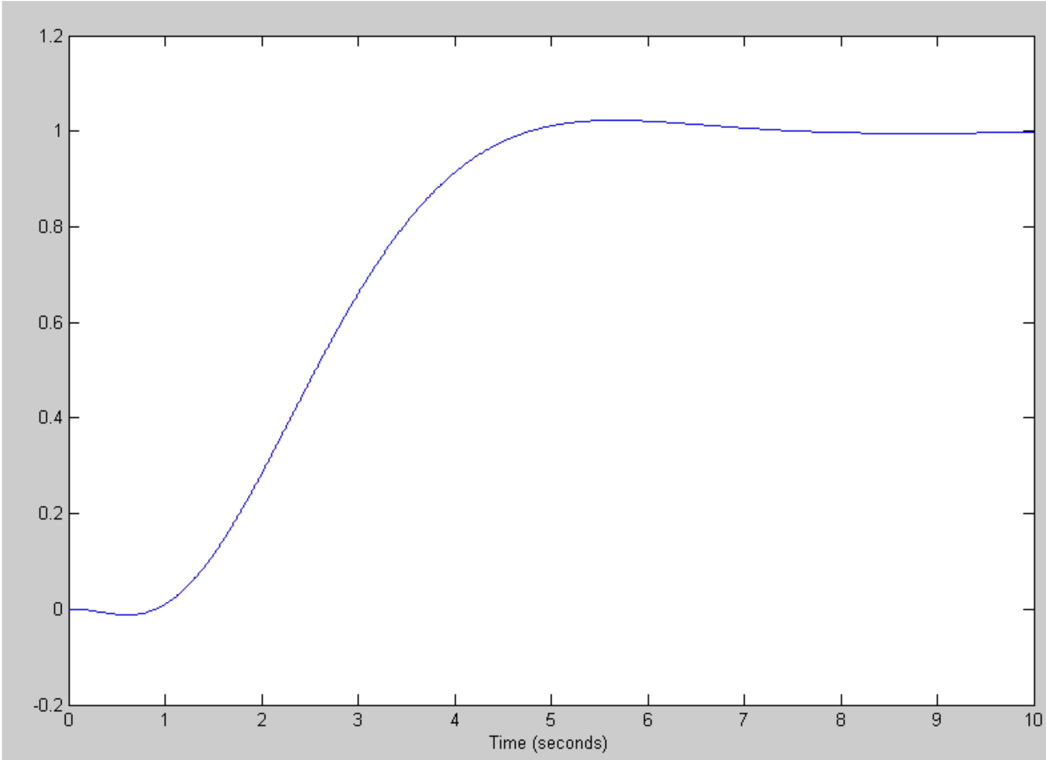
$n_x \sim N(0, 0.01^2)$ *mean zero, standard deviation 0.01*

$n_\theta \sim N(0, 0.01^2)$ *mean zero, standard deviation 0.01*

1) Use a servo-compensator to force the DC gain to one (i.e. use the servo compensator from homework set #10.

```
A = [0,0,1,0 ; 0,0,0,1 ; 0,-14.7,0,0 ; 0,24.5,0,0];
B = [0;0;0.5;-0.5];
C = [1,0,0,0];

A5 = [A,zeros(4,1) ; C, 0];
B5 = [B;0];
B5r = [0*B;-1];
C5 = [C,0];
D5 = 0;

K5 = lqr(A5, B5, diag([20,0,0,0,30]), 1);
G5 = ss(A5-B5*K5, B5r, C5, D5);
y = step(G5,t);
```

```
plot(t,y);
```



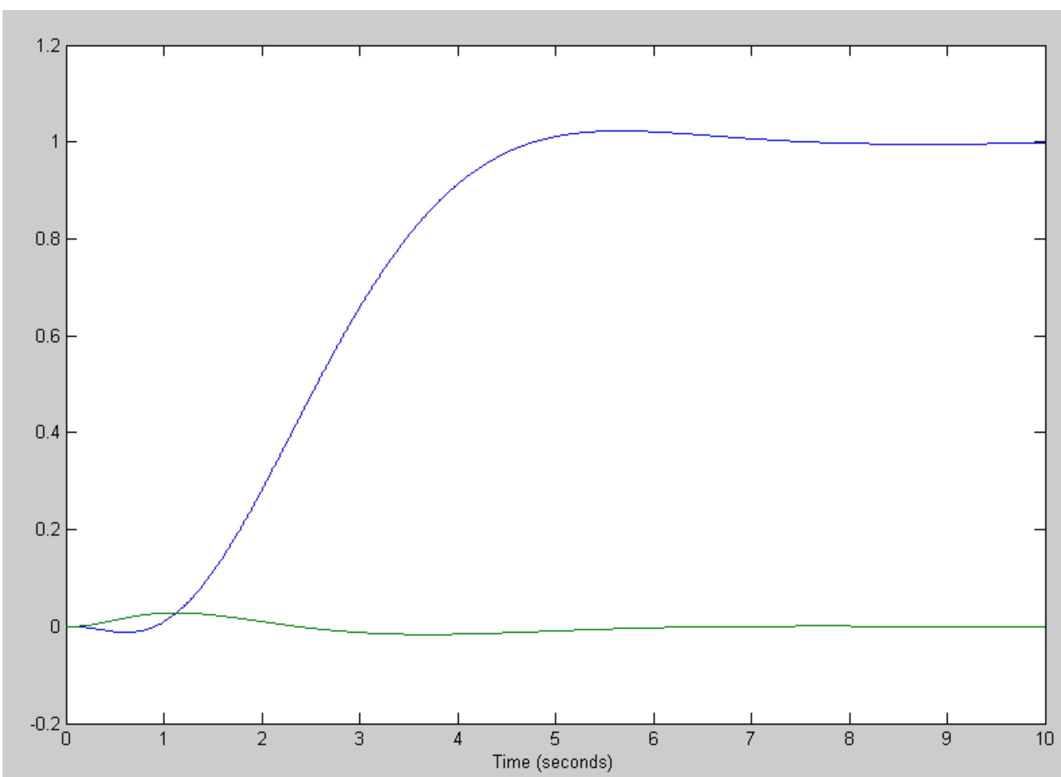
With noise, the dynamics are

$$\begin{bmatrix} sX \\ sZ \end{bmatrix} = \begin{bmatrix} A - BKx & -BKz \\ C & 0 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} R + \begin{bmatrix} B \\ 0 \end{bmatrix} N_u + \begin{bmatrix} BK_x \\ 0 \end{bmatrix} N_x + \begin{bmatrix} BK_q \\ 0 \end{bmatrix} N_q$$

or

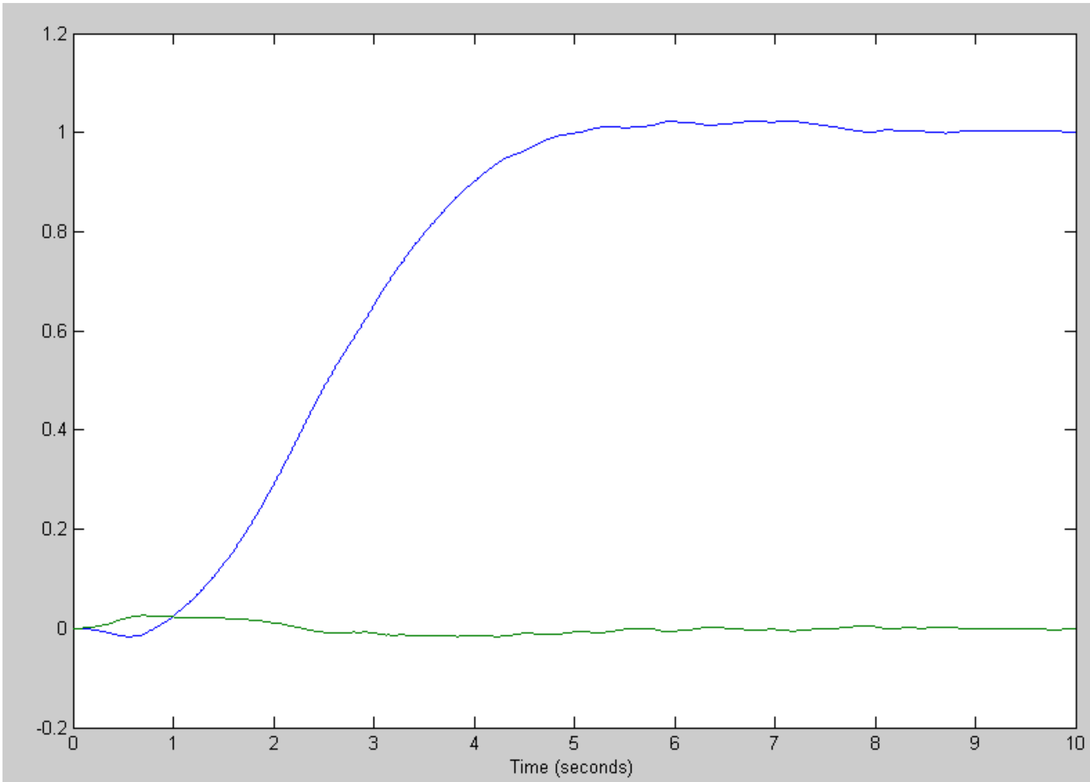
$$\begin{bmatrix} sX \\ sZ \end{bmatrix} = \begin{bmatrix} A - BKx & -BKz \\ C & 0 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} 0 & B & BK_x & BK_q \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ N_u \\ N_x \\ N_q \end{bmatrix}$$

```
>> Nu = randn(size(t)) * 0.02;
>> Nx = randn(size(t)) * 0.01;
>> Nq = randn(size(t)) * 0.01;
>> R  = 0*t + 1;

>> X0 = zeros(5,1);

>> A5 = [A,zeros(4,1) ; C, 0];
>> B5r = [0*B;-1];
>> B5u = [B;0];
>> B5x = [B*K5(1);1];
>> B5q = [B*K5(2);0];

>> [B5r,B5u,B5x,B5q]

      R          Nu         Nx       Nq
         0          0          0          0
         0          0          0          0
         0     0.5000    -7.1285   -80.4625
         0    -0.5000     7.1285    80.4625
   -1.0000          0     1.0000          0


>> t = [0:0.01:10]';
>> R = 0*t + 1;
>> y = step3(A5-B5*K5,[B5r,B5u,B5x,B5q],C5,D5,t,X0,[R,Nu*0,Nx*0,Nq*0]);
>> plot(t,y)
>> xlabel('Time (seconds)');
```



Step Response without Noise

```
>> y = step3(A5-B5*K5,[B5r,B5u,B5x,B5q],C5,D5,t,X0,[R,Nu,Nx,Nq]);
>> plot(t,y)
>> xlabel('Time (seconds)');
```



2) Design a full-order observer using pole-placement to place the observer poles at {-3, -4, -5, -6}

- Simulate the response of the cart with noise added at the input and output.
- Plot the states of the plant and the observer with noise,.

```
>> H = ppl(A', C', [-3,-4,-5,-6])'

H =

   18.0000
  -31.2245
  168.0000
 -219.1837
```

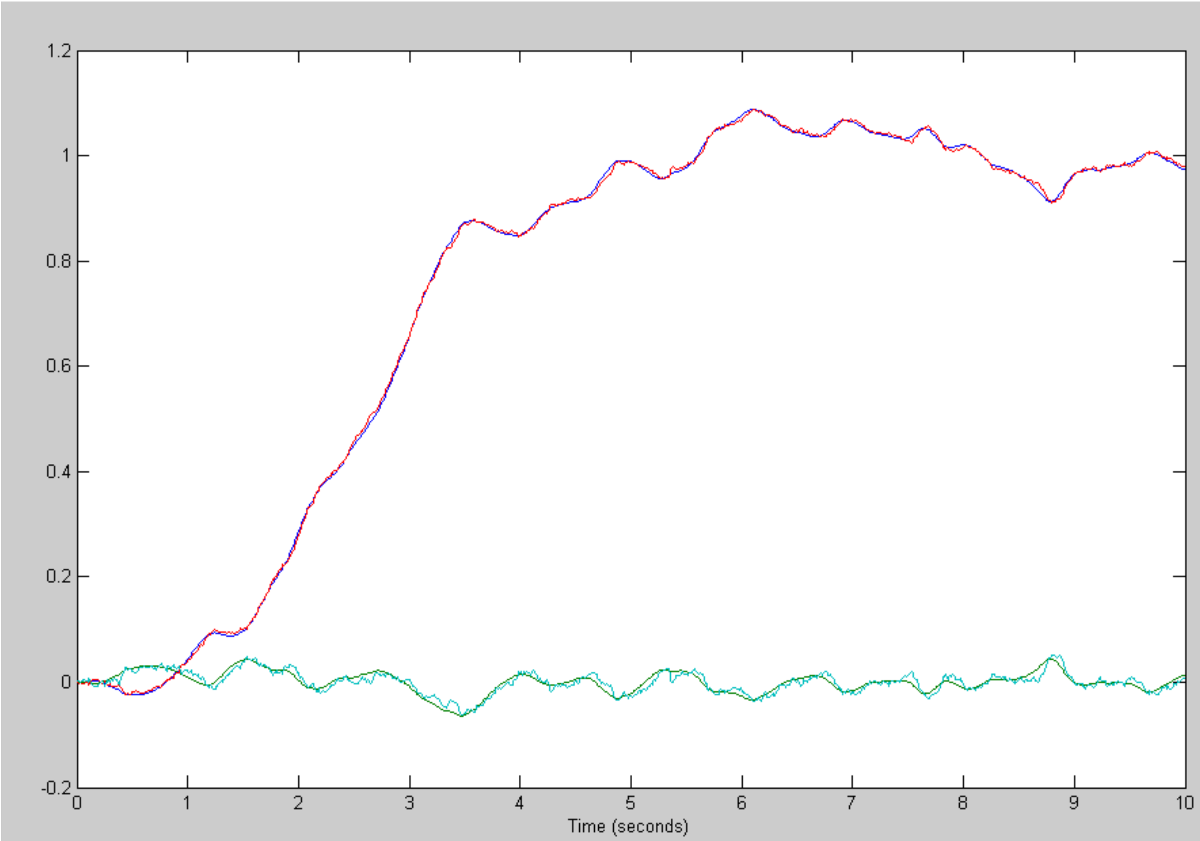
The plant + servo compensator + obsever is then

$$\begin{bmatrix} sX \\ sZ \\ sX_e \end{bmatrix} = \begin{bmatrix} A & -BK_z & -BK_x \\ C & 0 & 0 \\ HC & -BK_z & A-HC-BK_x \end{bmatrix} \begin{bmatrix} X \\ Z \\ X_e \end{bmatrix} + \begin{bmatrix} 0 & B & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & H_x & 0 \end{bmatrix} \begin{bmatrix} R \\ N_u \\ N_x \\ N_q \end{bmatrix}$$

Simulating in Matlab

```
>> C = [1,0,0,0]

C =      1     0     0     0

>> H = ppl(A', C', [-3,-4,-5,-6])'

   18.0000
  -31.2245
  168.0000
 -219.1837

>> A9 = [A, -B*Kz,-B*Kx ; C, 0, zeros(1,4) ; H*C, -B*Kz, A-H*C-B*Kx];

>> eig(A9)

 -19.6415
  -6.0000
  -5.0000
  -4.0000
  -3.0000
  -2.3498
  -0.6569 + 0.9153i
  -0.6569 - 0.9153i
  -0.9162

>> B9r = [0*B;-1;0*B];
>> B9u = [B;0;0*B];
>> B9x = [0*B;1;H];
>> B9q = [0*B;0;0*H];
```

```
>> B9 = [B9r,B9u,B9x,B9q]

      R          Nu        Nx           Nq
         0          0          0          0
         0          0          0          0
         0     1.0000          0          0
         0    -1.0000          0          0
   -1.0000          0     1.0000          0
         0          0    18.0000          0
         0          0   -31.2245          0
         0          0   168.0000          0
         0          0  -219.1837          0

>> X0 = zeros(9,1);
>> C9 =
[1,0,0,0,0,0,0,0,0;0,1,0,0,0,0,0,0,0;0,0,0,0,0,1,0,0,0;0,0,0,0,0,0,1,0,0]

C9 =

     1     0     0     0     0     0     0     0     0
     0     1     0     0     0     0     0     0     0
     0     0     0     0     0     1     0     0     0
     0     0     0     0     0     0     1     0     0

>> D9 = zeros(4,4);
>> y = step3(A9,B9,C9,D9,t,X0,[R,Nu,Nx,Nq]);
>> plot(t,y)
>> xlabel('Time (seconds)');
```



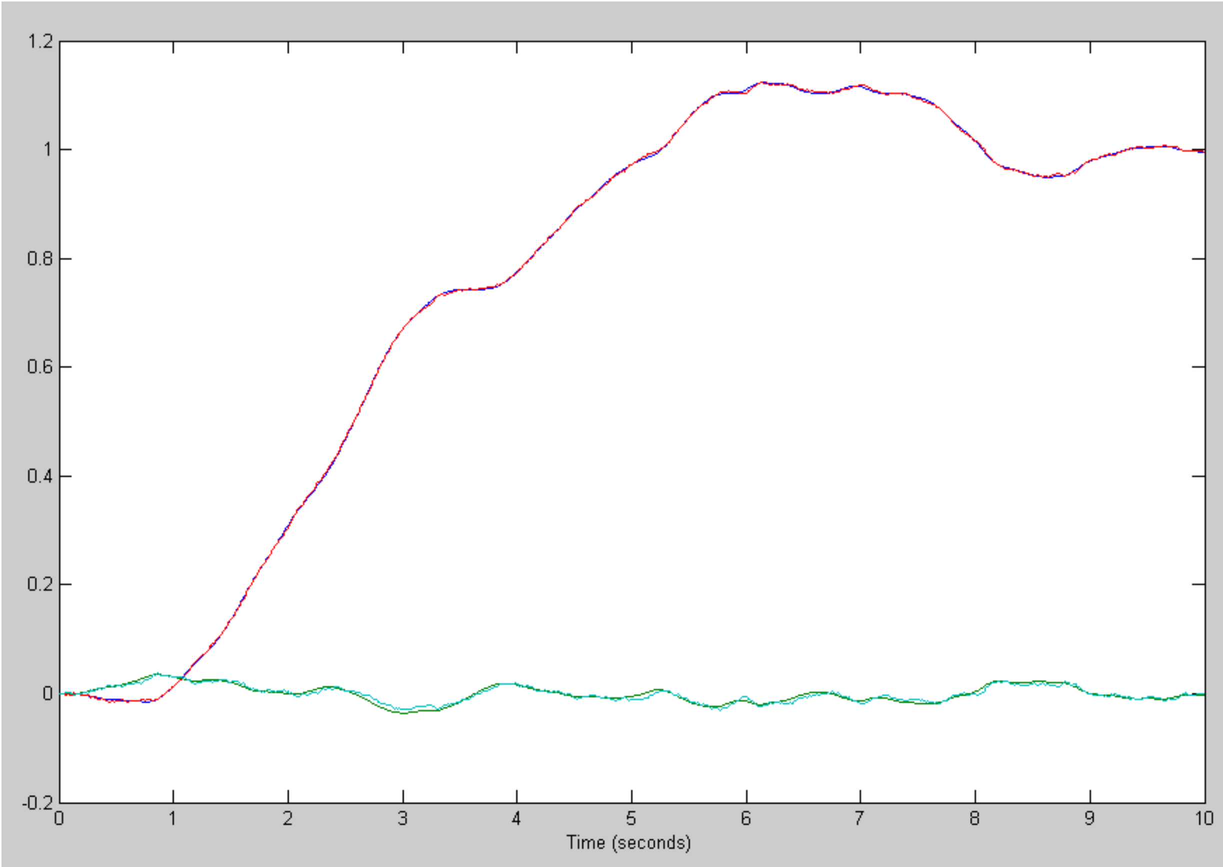
3) Design a Kalman filter (i.e. a full-order observer with a specific Q and R)
   - Simulate the response of the cart with noise added at the input and output.
   - Plot the states of the plant and the observer with noise,.

```
>> F = B

F =

     0
     0
     1
    -1

>> Q = F*F' * 0.02^2'

Q =

  1.0e-003 *

         0         0         0         0
         0         0         0         0
         0         0    0.4000   -0.4000
         0         0   -0.4000    0.4000

>> R = diag([0.01^2,0.01^2]);
>> C = [1,0,0,0;0,1,0,0]
```

```
     1     0     0     0
     0     1     0     0

>> H = lqr(A', C', Q, R)'

    5.5430   -6.5890
   -6.5890    9.3683
   37.0701  -52.1014
  -46.1495   65.5900

>> Hx = H(:,1);
>> Hq = H(:,2);
```

$$\begin{bmatrix} sX \\ sZ \\ sX_e \end{bmatrix} = \begin{bmatrix} A & -BK_z & -BK_x \\ C & 0 & 0 \\ HC & -BK_z & A-HC-BK_x \end{bmatrix} \begin{bmatrix} X \\ Z \\ X_e \end{bmatrix} + \begin{bmatrix} 0 & B & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & H_x & H_q \end{bmatrix} \begin{bmatrix} R \\ N_u \\ N_x \\ N_q \end{bmatrix}$$

```
>> B9x = [0*B;1;Hx];
>> B9q = [0*B;0;Hq];
>> B9q = [0*B;0;Hq];
>> B9 = [B9r,B9u,B9x,B9q]

         0         0         0         0
         0         0         0         0
         0    1.0000         0         0
         0   -1.0000         0         0
   -1.0000         0    1.0000         0
         0         0    5.5430   -6.5890
         0         0   -6.5890    9.3683
         0         0   37.0701  -52.1014
         0         0  -46.1495   65.5900

>> R = 0*t + 1;
>> y = step3(A9,B9,C9,D9,t,X0,[R,Nu,Nx,Nq]);
>> plot(t,y)
>> xlabel('Time (seconds)');
>>
```

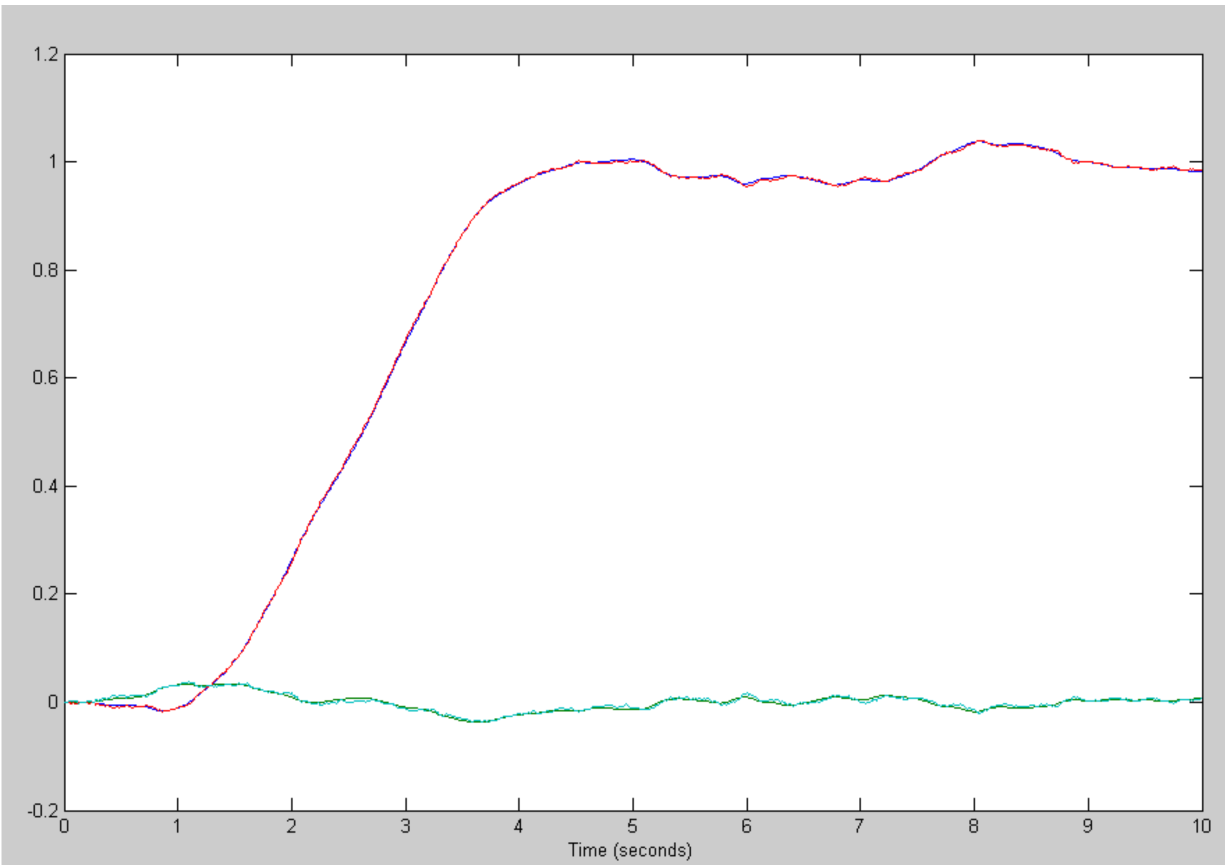


Step Respond with Noise: Servo Compensator Input is the Sensor (Yx)

If instead you use the observer estimte of x rather than x

$$\begin{bmatrix} sX \\ sZ \\ sX_e \end{bmatrix} = \begin{bmatrix} A & -BK_z & -BK_x \\ 0 & 0 & C_x \\ HC & -BK_z & A-HC-BK_x \end{bmatrix} \begin{bmatrix} X \\ Z \\ X_e \end{bmatrix} + \begin{bmatrix} 0 & B & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & H_x & H_q \end{bmatrix} \begin{bmatrix} R \\ N_u \\ N_x \\ N_q \end{bmatrix}$$

the step response is much cleaner (due to the less noise of the input to the servo compensator)



Step response with noise: servo copensator input is the state estimate of position