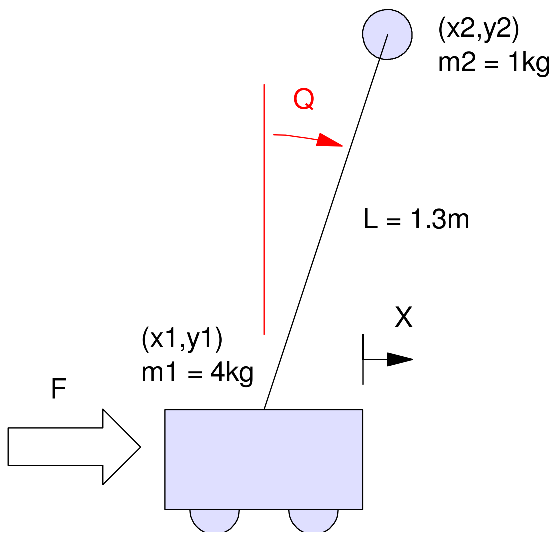
ECE 463/663 - Homework #4

LaGrangian Dynamics. Spring 2024

(30pt) Derive the dynamics for an inverted pendulum where

- m1 = 4kg (mass of cart)
- m2 = 1kg (mass of ball)
- L = 1.3m (length of arm)

Fine the linearized dynamics at x = 0, $\theta = 0$



Mass #1 (m1 = 4 kg)

$$x_1 = x \qquad y_1 = 0$$

$$\dot{x}_1 = \dot{x} \qquad \dot{y}_1 = 0$$

$$PE = 0$$

$$KE = \frac{1}{2}mv^2 = 2\dot{x}^2$$

Mass #2 (m2 = 1 kg, L = 1.3m)

$$x_2 = x + 1.3\sin\theta \qquad y_2 = 1.3\cos\theta$$

$$\dot{x}_2 = \dot{x} + 1.3\cos\theta\dot{\theta} \qquad \dot{y}_2 = -1.3\sin\theta\dot{\theta}$$

$$PE = m_2 g y_2 = 1 \cdot g \cdot 1.3\cos\theta$$

$$KE = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$KE = \tfrac{1}{2}\left(\left(\dot{x} + 1.3\cos\theta\dot{\theta}\right)^2 + \left(-1.3\sin\theta\dot{\theta}\right)^2\right)$$

$$KE = 0.5\dot{x}^2 + 0.845\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right) + 1.3\dot{x}\dot{\theta}\cos\theta$$

$$KE = 0.5\dot{x}^2 + 0.845\dot{\theta}^2 + 1.3\dot{x}\dot{\theta}\cos\theta$$

The LaGrangian is then

$$L = KE - PE$$

$$L = (2\dot{x}^2) + \left(0.5\dot{x}^2 + 0.845\dot{\theta}^2 + 1.3\dot{x}\dot{\theta}\cos\theta\right) - (1.3g\cos\theta)$$

To find the dynamics, use the Euler LaGrange equation

$$L = \left(2.5\dot{x}^2 + 0.845\dot{\theta}^2 + 1.3\dot{x}\dot{\theta}\cos\theta\right) - (1.3g\cos\theta)$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \tfrac{d}{dt}\left(5\dot{x} + 1.3\cos\theta\dot{\theta}\right) - (0)$$

$$\boxed{F = 5\ddot{x} + 1.3\cos\theta\ddot{\theta} - 1.3\sin\theta\dot{\theta}^2}$$

$$L = \left(2.5\dot{x}^2 + 0.845\dot{\theta}^2 + 1.3\dot{x}\dot{\theta}\cos\theta\right) - (1.3g\cos\theta)$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \tfrac{d}{dt}\left(1.69\dot{\theta} + 1.3\cos\theta\dot{x}\right) - \left(-1.3\sin\theta\dot{x}\dot{\theta} + 1.3g\sin\theta\right)$$

$$T = 1.69\ddot{\theta} + 1.3\cos\theta\ddot{x} - 1.3\sin\theta\dot{x}\dot{\theta} + 1.3\sin\theta\dot{x}\dot{\theta} - 1.3g\sin\theta$$

$$\boxed{T = 1.69\ddot{\theta} + 1.3\cos\theta\ddot{x} - 1.3g\sin\theta}$$

So, the dynamics are

$$F = 5\ddot{x} + 1.3\cos\theta\ddot{\theta} - 1.3\sin\theta\dot{\theta}^2$$

$$T = 1.69\ddot{\theta} + 1.3\cos\theta\ddot{x} - 1.3g\sin\theta$$

In Matrix form

$$\begin{bmatrix} 5 & 1.3\cos\theta \\ 1.3\cos\theta & 1.69 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F \\ T \end{bmatrix} + \begin{bmatrix} 1.3\sin\theta\dot{\theta}^2 \\ 1.3g\sin\theta \end{bmatrix}$$

Linearizing about zero with T = 0

$$\begin{bmatrix} 5 & 1.3 \\ 1.3 & 1.69 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F + \begin{bmatrix} 0 \\ 1.3g\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0.25 \\ -0.1923 \end{bmatrix} F + \begin{bmatrix} -0.25g\theta \\ 0.9615g\theta \end{bmatrix}$$

Putting this in state-space form

$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2.45 & 0 & 0 \\ 0 & 9.42 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.25 \\ -0.1923 \end{bmatrix} F$$

The poles are at

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>> A = [0,0,1,0;0,0,0,1;0,-2.45,0,0;0,9.42,0,0]

         0         0    1.0000         0
         0         0         0    1.0000
         0   -2.4500         0         0
         0    9.4200         0         0

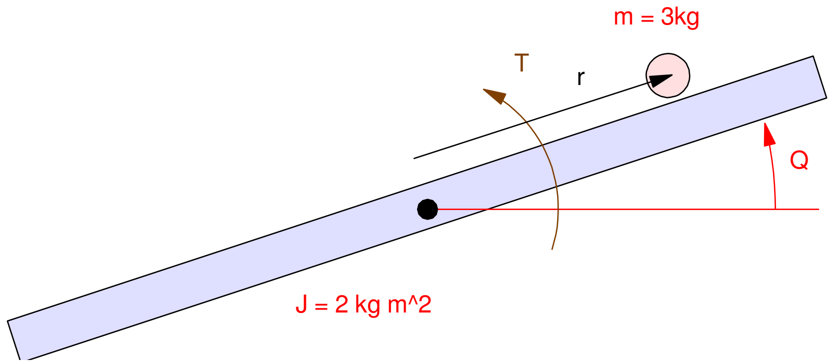
>> eig(A)

         0
         0
    3.0692
   -3.0692
```

2) (30pt) Derive the dynamics for a ball and beam system where

- J = 2 kg m2 (the inertia of the beam)
- m = 3 kg (the mass of the ball)

Find the linearized dynamics at r = 1.0m, θ = 0



Position of the ball:

$$x_1 = r\cos\theta \qquad y_1 = r\sin\theta$$

$$\dot{x}_1 = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \qquad \dot{y}_1 = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$$

The potential and kinetic energy. Assuming a solid sphere with radius 5mm (0.005m)

$$J = \tfrac{2}{5}mr^2 \qquad \textit{r = radius here}$$

$$x = r\theta \qquad \textit{x = displacement here}$$

$$\dot{x} = r\dot{\theta}$$

$$KE = \tfrac{1}{2}J\dot{\theta}^2 = \tfrac{1}{2}\left(\tfrac{2}{5}mr^2\right)\left(\tfrac{\dot{x}}{r}\right)^2 = \tfrac{1}{5}m\dot{x}^2 \qquad \textit{rotational KE for a solid sphere rolling}$$

This gives (using r for displacement along the beam for x)

$$PE = mgy_1 = mgr\sin\theta = 3gr\sin\theta$$

$$KE = \tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \tfrac{1}{5}m\dot{r}^2$$

$$KE = \dot{\theta}^2 + 1.5\left(\dot{x}_1^2 + \dot{y}_1^2\right) + 0.6\dot{r}^2$$

$$KE = \dot{\theta}^2 + 1.5\left(\left(\dot{r}\cos\theta - r\sin\theta\dot{\theta}\right)^2 + \left(\dot{r}\sin\theta + r\cos\theta\dot{\theta}\right)^2\right) + 0.6\dot{r}^2$$

$$KE = \dot{\theta}^2 + 1.5\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + 0.6\dot{r}^2$$

$$KE = (1 + 1.5r^2)\dot{\theta}^2 + 2.1\dot{r}^2$$

The LaGrangian is then

$$L = KE - PE$$

$$L = \left((1+1.5r^2)\dot{\theta}^2 + 2.1\dot{r}^2\right) - (3gr\sin\theta)$$

$$L = \left(\dot{\theta}^2 + 1.5r^2\dot{\theta}^2 + 2.1\dot{r}^2\right) - (3gr\sin\theta)$$

Force on the Ball

$$L = \left(\dot{\theta}^2 + 1.5r^2\dot{\theta}^2 + 2.1\dot{r}^2\right) - (3gr\sin\theta)$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right)$$

$$F = \frac{d}{dt}(4.2\dot{r}) - \left(3r\dot{\theta}^2 - 3g\sin\theta\right)$$

$$\boxed{F = 4.2\ddot{r} - 3r\dot{\theta}^2 + 3g\sin\theta}$$

Torque on the Beam

$$L = \left(\dot{\theta}^2 + 1.5r^2\dot{\theta}^2 + 2.1\dot{r}^2\right) - (3gr\sin\theta)$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \frac{d}{dt}\left(2\dot{\theta} + 3r^2\dot{\theta}\right) - (-3gr\cos\theta)$$

$$\boxed{T = 2\ddot{\theta} + 3r^2\ddot{\theta} + 6r\dot{r}\dot{\theta} + 3gr\cos\theta}$$

Putting it together

$$\begin{bmatrix} 4.2 & 0 \\ 0 & 2+3r^2 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 3r\dot{\theta}^2 - 3g\sin\theta \\ -6r\dot{r}\dot{\theta} - 3gr\cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}T$$

Linearizing at r = 1.0m

$$\begin{bmatrix} 4.2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -3g\theta \\ -3gr \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}T$$

In State-Space form

$$s\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -7 & 0 & 0 \\ -5.88 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.2 \end{bmatrix} T$$

The open-loop system is unstable

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>> A = [0,0,1,0;0,0,0,1;0,-7,0,0;-5.88,0,0,0]

         0         0    1.0000         0
         0         0         0    1.0000
         0   -7.0000         0         0
   -5.8800         0         0         0

>> eig(A)

  -2.5329
  -0.0000 + 2.5329i
  -0.0000 - 2.5329i
   2.5329

>>
```