ECE 463/663 - Homework #12

LQR Observers. Due Monday, April 22nd

Kalman Filters

Cart and Pendulum (HW #4): The dynamics for a cart and pendulum system with sensor and input noise is as follows

$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2.45 & 0 & 0 \\ 0 & 9.42 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.25 \\ -0.1923 \end{bmatrix} (F + \eta_u)$$

$$y_1 = x + n_x$$

$$y_2 = \theta + n_\theta$$

where there is Gaussian noise at the input and output

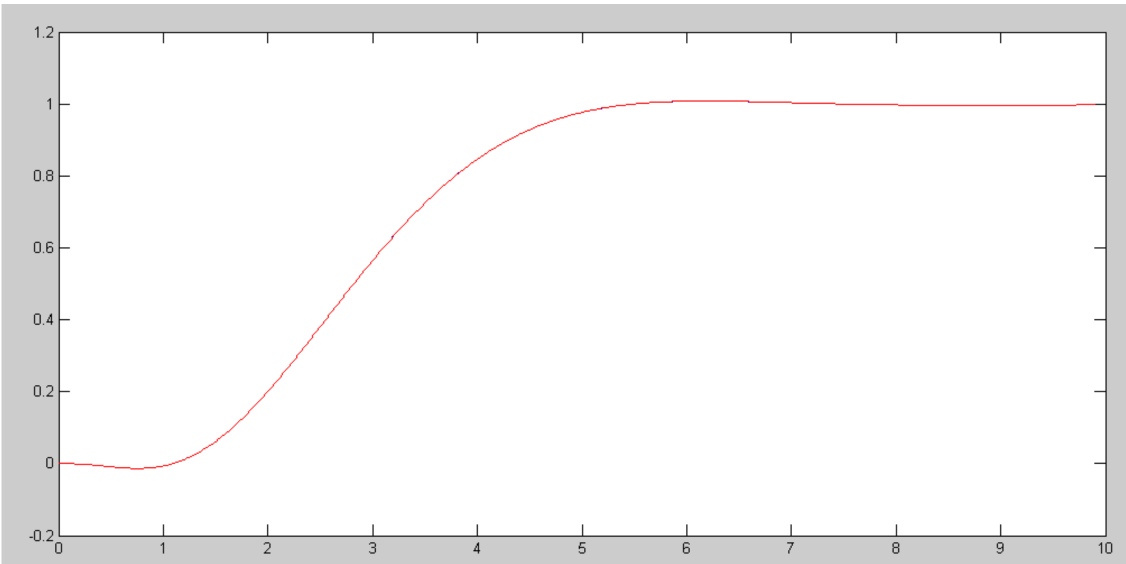
$n_u \sim N(0, 0.5^2)$ *mean zero, standard deviation 0.5*

$n_x \sim N(0, 0.1^2)$ *mean zero, standard deviation 0.1*

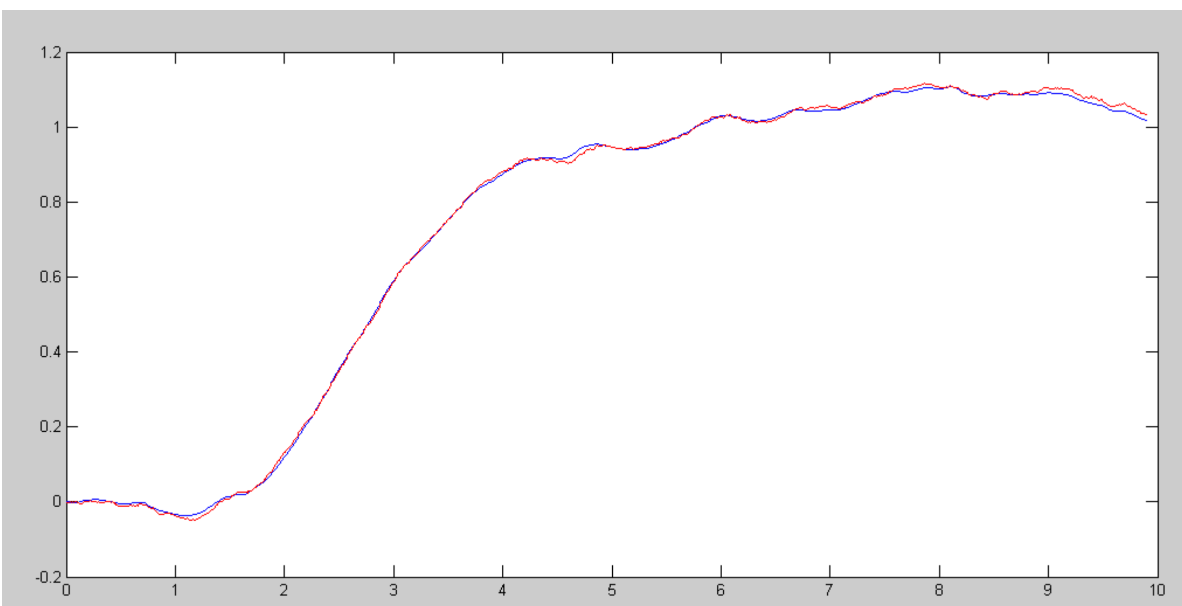
$n_\theta \sim N(0, 0.05^2)$ *mean zero, standard deviation 0.05*

1) Use a servo-compensator to force the DC gain to one (i.e. use the servo compensator from homework set #10.

No Noise



Plus Noise



Code:

```
% Cart and Pendulum ( Sp24 version)
% m1 = 4.0kg
% m2 = 1.0kg
% L = 1.3m

X = [0; 0 ; 0 ; 0];
Ref = 1;
dt = 0.01;
t = 0;
Kx = [-15.9909 -203.1194  -20.6044  -70.3334];
Kz = -5.4772;
Z = 0;


y = [];
while(t < 9.9)
 Ref = 1;

 nu = 0.5*randn;  %0.5
 nx = 0.1*randn;   %0.1
 nq = 0.05*randn   %0.05

 Y = X(1) + nx;
 U = - Kx*(X + [nx;nq;0;0]) - Kz*Z;

 dX = CartDynamics(X, U + nu);
 dZ = Y - Ref;

 X = X + dX * dt;
 Z = Z + dZ * dt;
 t = t + dt;

 CartDisplay(X, Xe, Ref);
 y = [y ; X(1), X(2), Xe(1), Xe(2), Ref];
end

clf
t = [1:length(y)]' * dt;
plot(t,y(:,1),'b',t,y(:,3),'r');
```

2) Design a full-order observer using pole-placement to place the observer poles at {-3, -3, -3, -3}

- Simulate the response of the cart with noise added at the input and output.
- Plot the states of the plant and the observer with noise,.

Note: Let

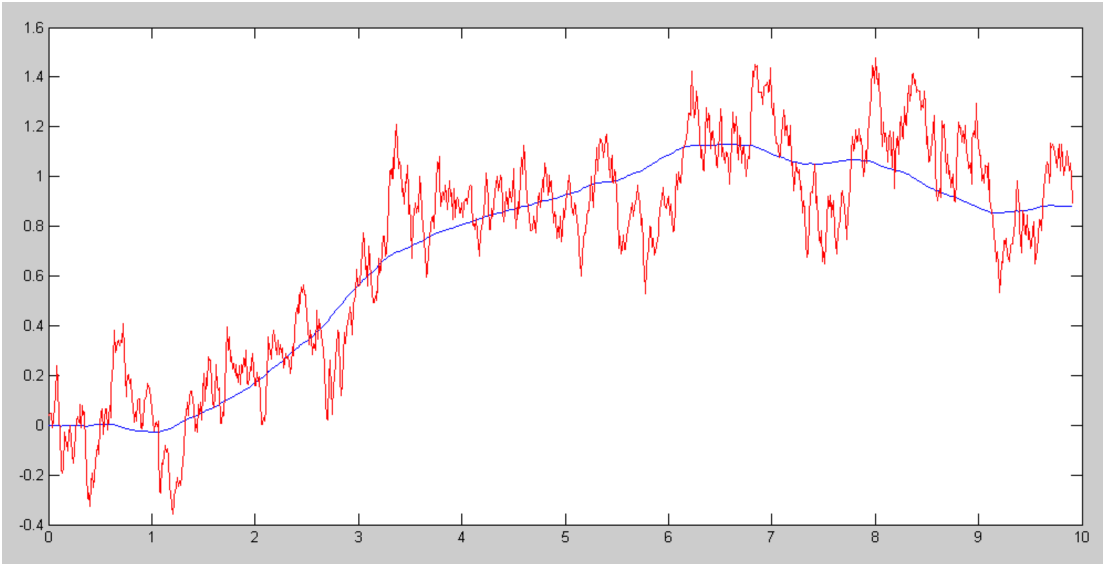
$$C = [1, 10, 0, 0]$$

so that I'm using position and angle.

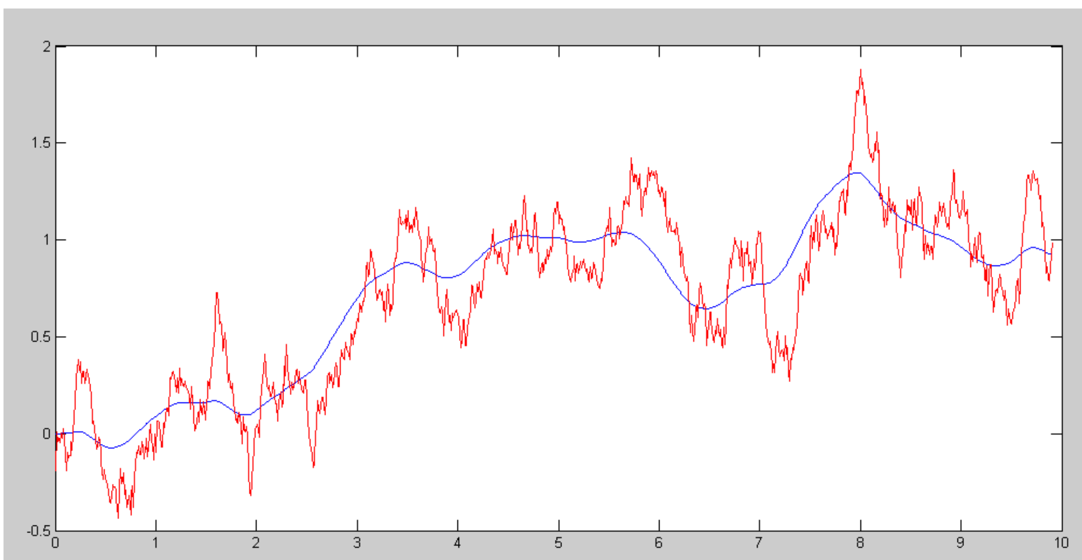
```
>> A = [0,0,1,0;0,0,0,1;0,-2.45,0,0;0,9.42,0,0];
>> B = [0;0;0.25;-0.1923];
>> C = [1,10,0,0];
>> H = ppl(A', C', [-3,-3,-3,-3])'

  -12.0916
    2.4092
  -10.5218
    7.3942
```

Effectively, H is a 4x2 matrix with the second column (times angle) 10x the first column

```
>> H = [H, 10*H]

  -12.0916 -120.9155
    2.4092   24.0916
  -10.5218 -105.2184
    7.3942   73.9418

>> C = [1,0,0,0;0,1,0,0]

     1     0     0     0
     0     1     0     0

>> eig(A - H*C)

  -3.0000 + 0.0004i
  -3.0000 - 0.0004i
  -2.9996
  -3.0004
```



Feedback using actual states (plus noise)



Feeding back the observer states

Comment: The large gains in H are messing up the observer's estimates

Code:

```
% Cart and Pendulum ( Sp24 version)
% m1 = 4.0kg
% m2 = 1.0kg
% L = 1.3m

X = [0; 0 ; 0 ; 0];
Ref = 1;
dt = 0.01;
t = 0;
Kx = [-15.9909 -203.1194  -20.6044  -70.3334];
Kz = -5.4772;
Z = 0;


% Observer
A = [0,0,1,0;0,0,0,1;0,-2.45,0,0;0,9.42,0,0];
B = [0;0;0.25;-0.1923];
C = [1,0,0,0 ; 0,1,0,0];
Xe = X;
Q = B*B' * 0.5^2;
R = diag([0.1^2, 0.05^2]);
%H = lqr(A', C', Q, R)';
C = [1,10,0,0];
H = ppl(A', C', [-3, -3, -3, -3])';

y = [];
while(t < 9.9)
 Ref = 1;

 nu = 0.5*randn;
 nx = 0.1*randn;
 nq = 0.05*randn;

 Y = X(1) + nx;
 %U = - Kx*(X + [nx;nq;0;0]) - Kz*Z;
 U = - Kx*Xe - Kz*Z;

 dX = CartDynamics(X, U + nu);
 dZ = Y - Ref;
 dXe = A*Xe + B*U + H*(C*(X + [nx;nq;0;0]) - C*Xe);

 X = X + dX * dt;
 Xe = Xe + dXe * dt;
 Z = Z + dZ * dt;
 t = t + dt;

 CartDisplay(X, Xe, Ref);
 y = [y ; X(1), X(2), Xe(1), Xe(2), Ref];
end

clf
t = [1:length(y)]' * dt;
plot(t,y(:,1),'b',t,y(:,3),'r');
```

3) Design a Kalman filter (i.e. a full-order observer with a specific Q and R)

- Simulate the response of the cart with noise added at the input and output.
- Plot the states of the plant and the observer with noise,.

```
% Observer
A = [0,0,1,0;0,0,0,1;0,-2.45,0,0;0,9.42,0,0];
B = [0;0;0.25;-0.1923];
C = [1,0,0,0 ; 0,1,0,0];
Xe = X;
Q = B*B' * 0.5^2;
R = diag([0.1^2, 0.05^2]);
H = lqr(A', C', Q, R)';

H (LQR)

    1.4715   -1.1492
   -0.2873    6.1120
    1.2477   -4.9047
   -0.9526   18.8436
```

Note: The Kalman filter uses smaller gains (doesn't amplify the noise as much

```
H (pole-placement)

  -12.0916 -120.9155
    2.4092   24.0916
  -10.5218 -105.2184
    7.3942   73.9418
```

Smaller gains keep the noise levels down

Sidelight: With the full-order observer, the servo compensator could

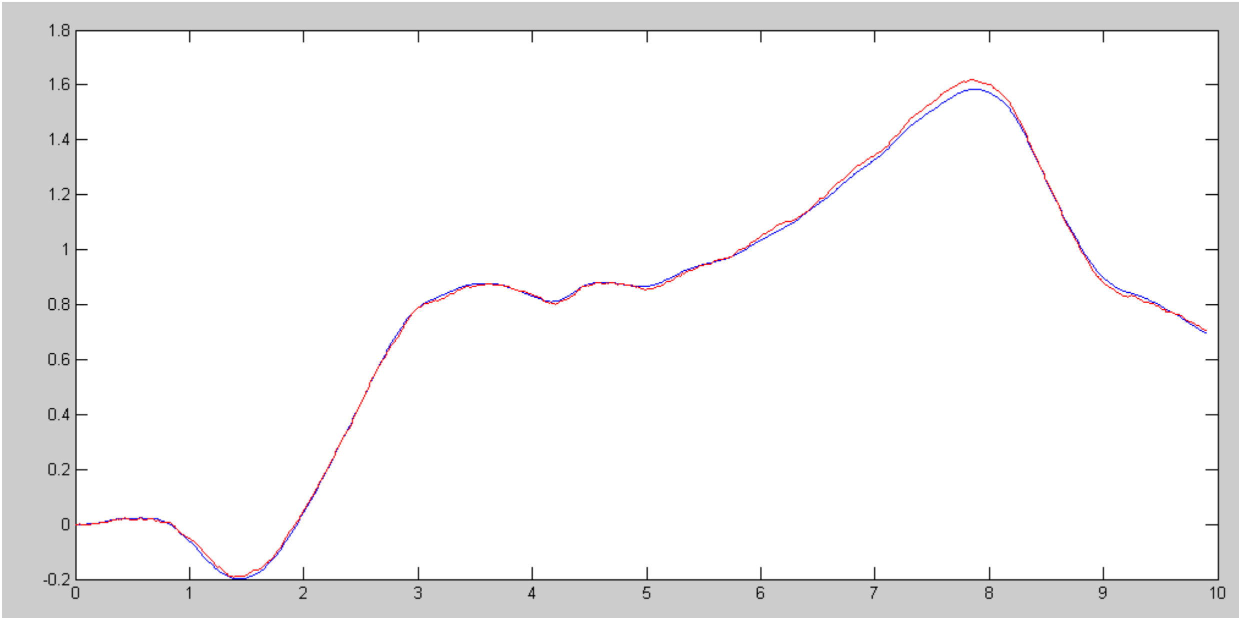
- Use the actual output, or

  $$sZ = Y - Ref$$

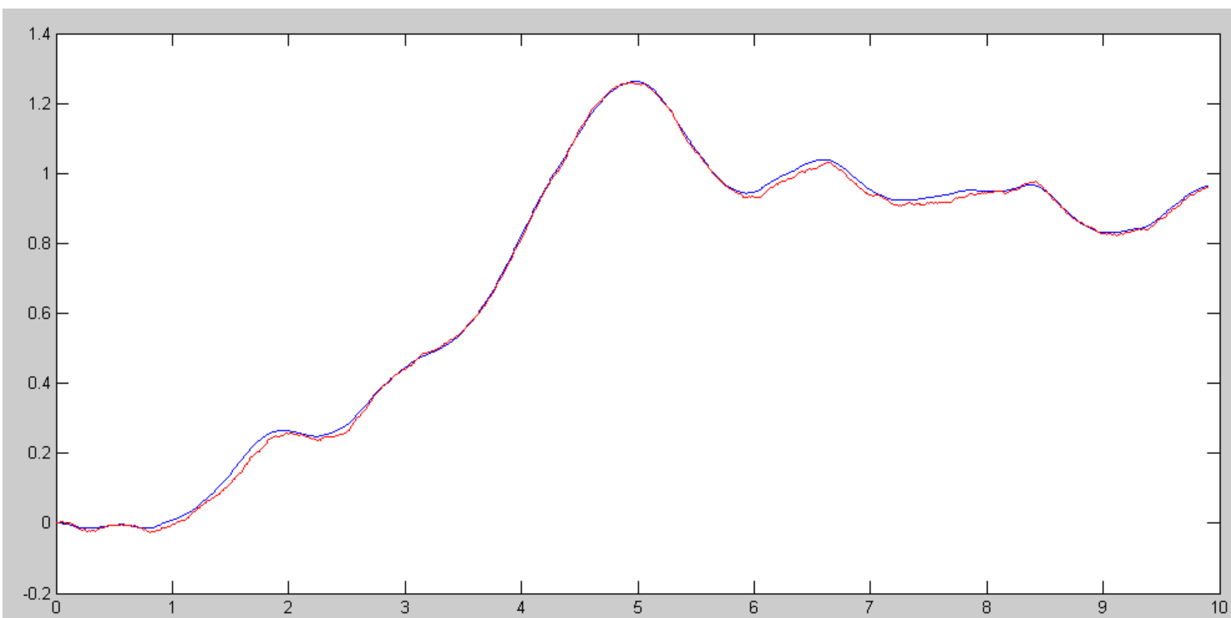
- The observer's output

  $$sZ = Y_e - Ref$$

The latter actually has less noise and slightly beter tracking



dZ = (Y - Ref)



dZ = Xe(1) - Ref

Code:

```
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Kz = -5.4772;
Z = 0;


% Observer
A = [0,0,1,0;0,0,0,1;0,-2.45,0,0;0,9.42,0,0];
B = [0;0;0.25;-0.1923];
C = [1,0,0,0 ; 0,1,0,0];
Xe = X;
Q = B*B' * 0.5^2;
R = diag([0.1^2, 0.05^2]);
H = lqr(A', C', Q, R)';
%C = [1,10,0,0];
%H = ppl(A', C', [-3, -3, -3, -3])';

y = [];
while(t < 9.9)
 Ref = 1;

 nu = 0.5*randn;
 nx = 0.1*randn;
 nq = 0.05*randn;

 Y = X(1) + nx;
 U = - Kx*Xe - Kz*Z;

 dX = CartDynamics(X, U + nu);
 dZ = Y - Ref;
 dXe = A*Xe + B*U + H*(C*(X + [nx;nq;0;0]) - C*Xe);

 X = X + dX * dt;
 Xe = Xe + dXe * dt;
 Z = Z + dZ * dt;
 t = t + dt;

 CartDisplay(X, Xe, Ref);
 y = [y ; X(1), X(2), Xe(1), Xe(2), Ref];
end

clf
t = [1:length(y)]' * dt;
plot(t,y(:,1),'b',t,y(:,3),'r');
```