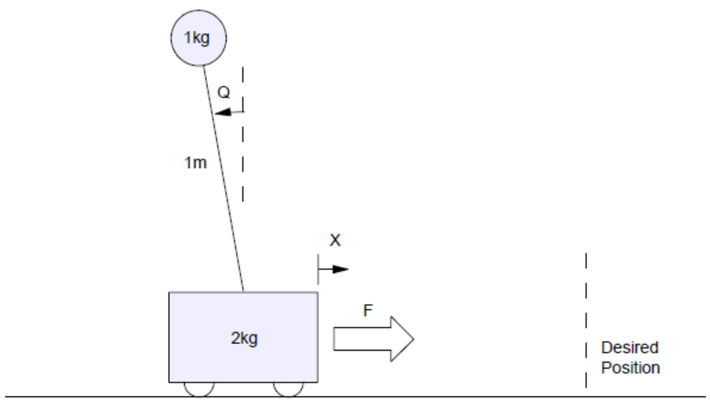
Pole Placement for a Cart and Pendulum

System:

Find the dynamics for the following system: A 1kg mass swings from a gantry which weighs 2kg. The length of the rope is 1m.



Nonlinear Model

From before, the nonlinear dynamics are:

$$\begin{bmatrix} 3 & 2\cos\theta \\ \cos\theta & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 2\dot{\theta}^2 \sin\theta \\ g\sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

Linear Model:

For small perturbations about zero,

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1$$

$$\dot{\theta}^2 \approx 0$$

$$g = 9.8 \frac{m}{s^2}$$

This results in the following state-space model:

$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -19.4 & 0 & 0 \\ 0 & 29.4 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}F$$

Pole Placement:

For a 2% settling time of 4 seconds and no overshoot, the dominant pole should be at -1. Somewhat arbitrarily, place the four poles at {-1, -2, -3, -4}. Using pole-placement methods, this results in Kx:

Matlab Code:

First, input the system

```
 >> A = [0,0,1,0;  0,0,0,1;  0,-19.4,0,0;  0,29.4,0,0]

         0         0    1.0000         0
         0         0         0    1.0000
         0  -19.4000         0         0
         0   29.4000         0         0

>> B = [0;0;1;-1]

     0
     0
     1
    -1

>> C = [1,0,0,0];
```

Check that gravity is in the correct direction (the system should be unstable)

```
>> eig(A)

         0
         0
    5.4222
   -5.4222
```

Compute the feedback gains to place the closed-loop poles at {-1, -2, -3, -4}

```
>> Kx = ppl(A, B, [-1, -2, -3, -4])

   -2.4000  -66.8000   -5.0000  -15.0000

>> eig(A - B*Kx)

   -4.0000
   -3.0000
   -1.0000
   -2.0000
```

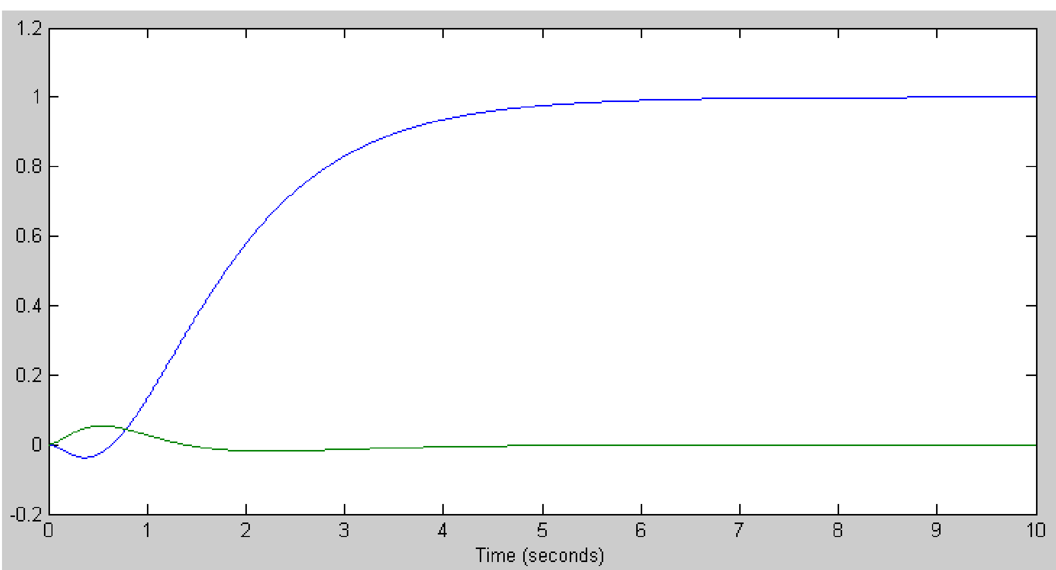
Check - Kx is correct. Now find Kr to make the DC gain one:

```
>> DC = -C*inv(A - B*Kx)*B

   -0.4167

>> Kr = 1/DC

   -2.4000
```

Done. Just for fun, plot the step response to

- Position (x), and
- Angle (θ)

```
>> C2 = [1,0,0,0; 0,1,0,0]

     1     0     0     0         position
     0     1     0     0         angle

>> D2 = [0;0]

     0
     0

>> G = ss(A-B*Kx, B*Kr, C2, D2);
>> t = [0:0.001:10]';
>> y = step(G,t);
>> plot(t,y)
>> xlabel('Time (seconds)');
```



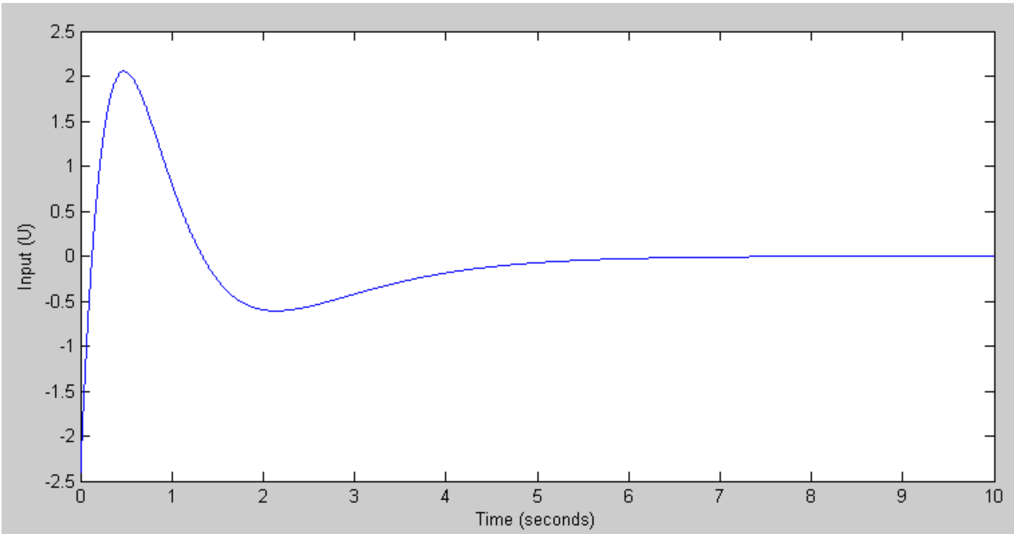
The input can be found by defining the output to be

$$Y = U = K_r R - K_x X$$

$$C = -K_x$$

$$D = K_r$$

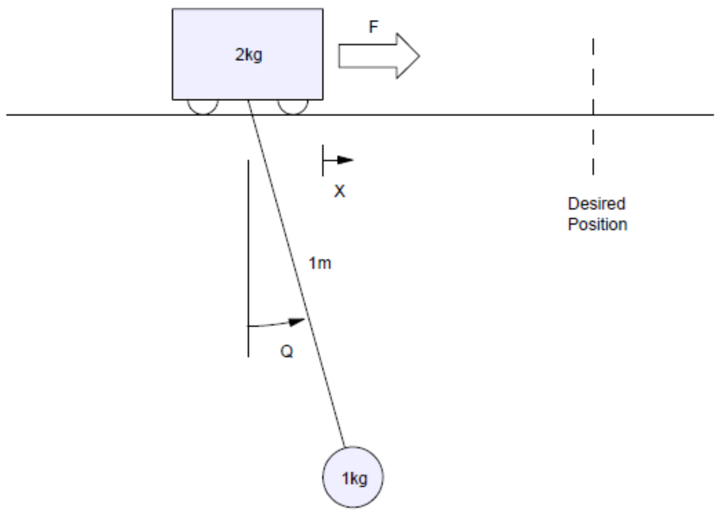
```
>> Gu = ss(A-B*Kx, B*Kr, -Kx, Kr);
>>
>> U = step(Gu,t);
>> plot(t,U);
>> xlabel('Time (seconds)');
>> ylabel('Input (U)');
```



Input vs. Time for the Closed Loop System with Poles at {-1, -2, -3, -4}

Example 2: Gantry System

If you flip the system (meaning change the sign of gravity), you get a gantry system:



The dynamics for this configuration are the same except g = -9.8 m/s2, resulting in

$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -19.4 & 0 & 0 \\ 0 & 29.4 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} F$$

Zero angle means point straight down (which changes the animation a little).

Finding the feedback gains to place the closed-loop poles at { -1, -2, -3, -4 }

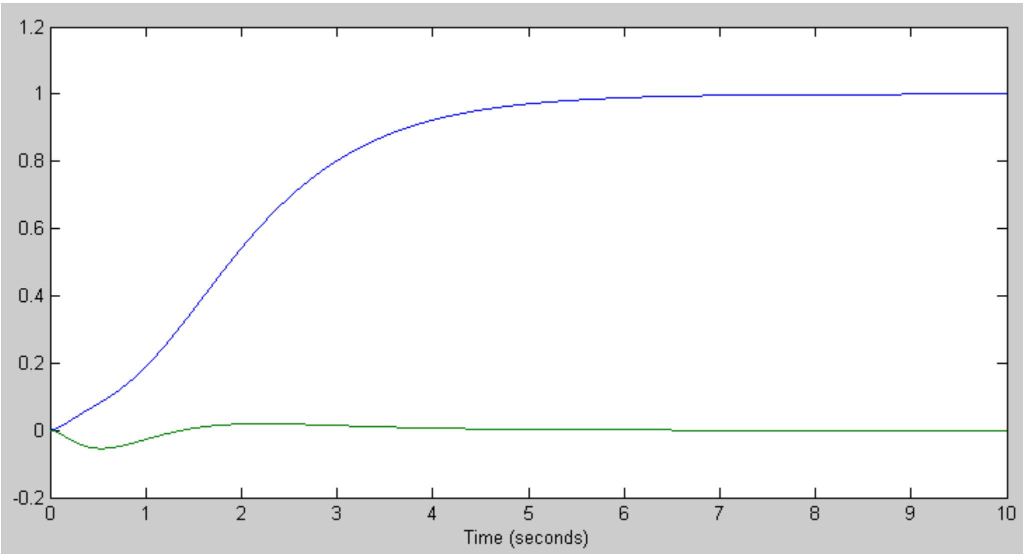
```
>> Kx = ppl(A, B, [-1, -2, -3, -4])

    2.4000   -3.2000    5.0000   -5.0000

>> eig(A - B*Kx)

   -4.0000
   -3.0000
   -1.0000
   -2.0000
```

Now find Kr to set the DC gain.

```
>> DC = -C*inv(A - B*Kx)*B

    0.4167

>> Kr = 1/DC

    2.4000
```

Plot the step response to position (x). Just for fun, also plot the step response to angle.

```
>> C2 = [1,0,0,0; 0,1,0,0]


C2 =

     1     0     0     0       position (x)
     0     1     0     0       angle (θ)

>> D2 = [0;0]

     0
     0

>> G = ss(A-B*Kx, B*Kr, C2, D2);
>> t = [0:0.001:10]';
>> y = step(G,t);
>> plot(t,y)
>> xlabel('Time (seconds)');
>>
```

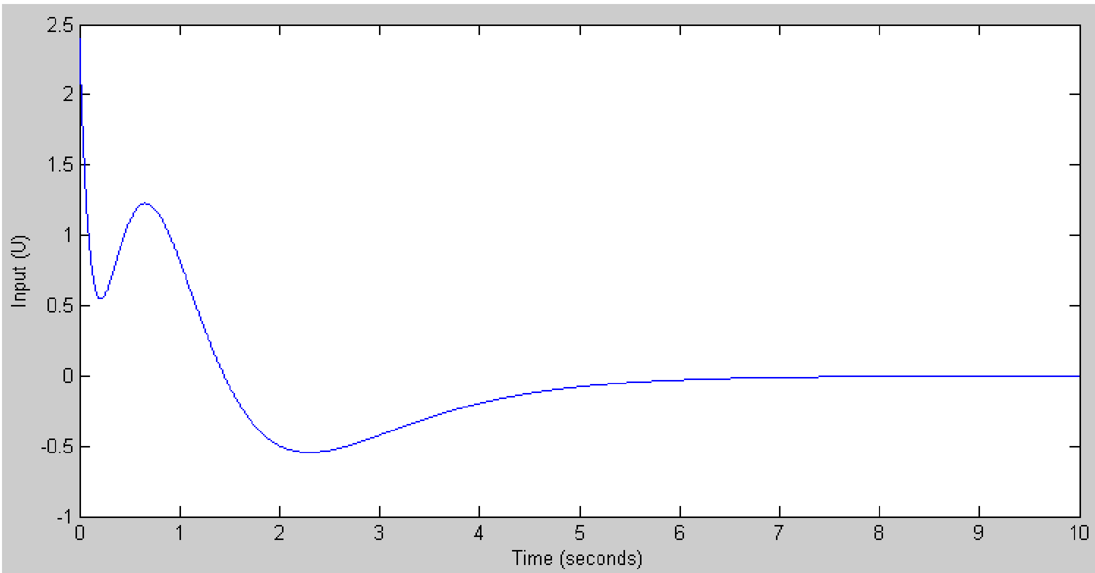


Step Response of a Gantry System with Poles Placed at {-1, -2, -3, -4}. Position = Blue, Angle = Green

The input can also be plotted using

$$U = -K_x X + K_r R$$

```
>> Gu = ss(A-B*Kx, B*Kr, -Kx, Kr);
>> U = step(Gu,t);
>> plot(t,U);
>> xlabel('Time (seconds)');
>> ylabel('Input (U)');
```



Input for the Closed-Loop System with Poles Placed at { -1, -2, -3, -4}