Optimal Control and the Ricatti Equation

NDSU ECE 463/663

Lecture #24

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Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

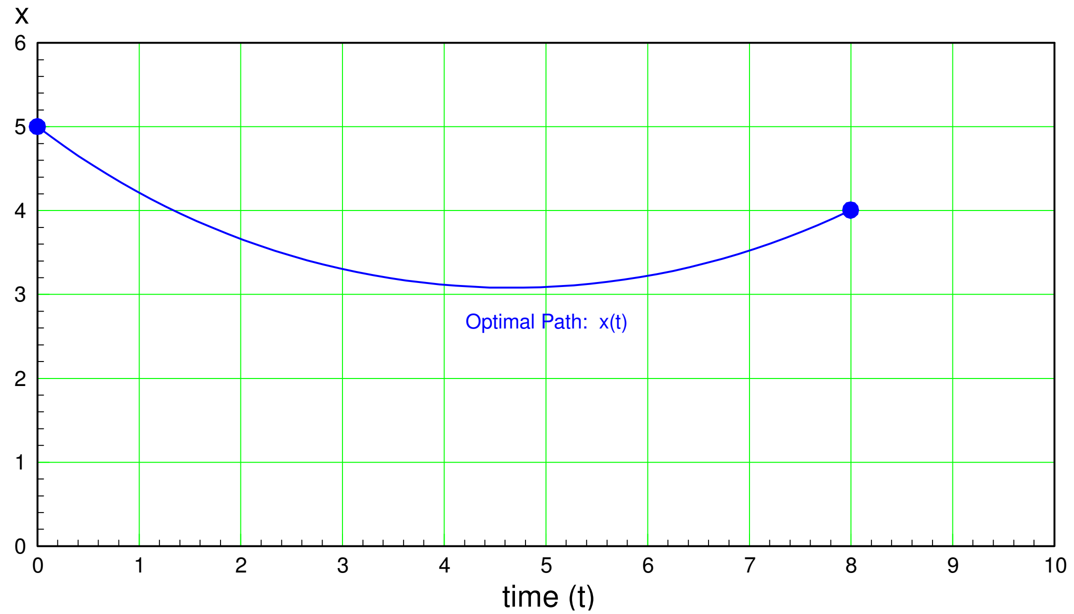
Calculus of Variations with Dynamic Systems

To minimize

$$J(x) = \int_a^b F(t, x, \dot{x})dt$$

x(t) must satisfy the Euler Legrange equation

$$F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$$



Example 1:

Find x(t) which minimizes

$$J = \int_0^8 (x^2 + \dot{x}^2)dt$$

Constraints:

- x(0) = 5
- x(8) = 4

Solution: The Euler Legrange equation gives

$$F = x^2 + \dot{x}^2$$

$$F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$$

$$2x - \frac{d}{dt}(2\dot{x}) = 0$$

$$x - \ddot{x} = 0$$

Using LaPlace notation

$$(1 - s^2)x = 0$$

Either

- x = 0 (the trivial solution) or
- s = {+1, -1}

The general solution is then

$$x(t) = ae^{t} + be^{-t}$$

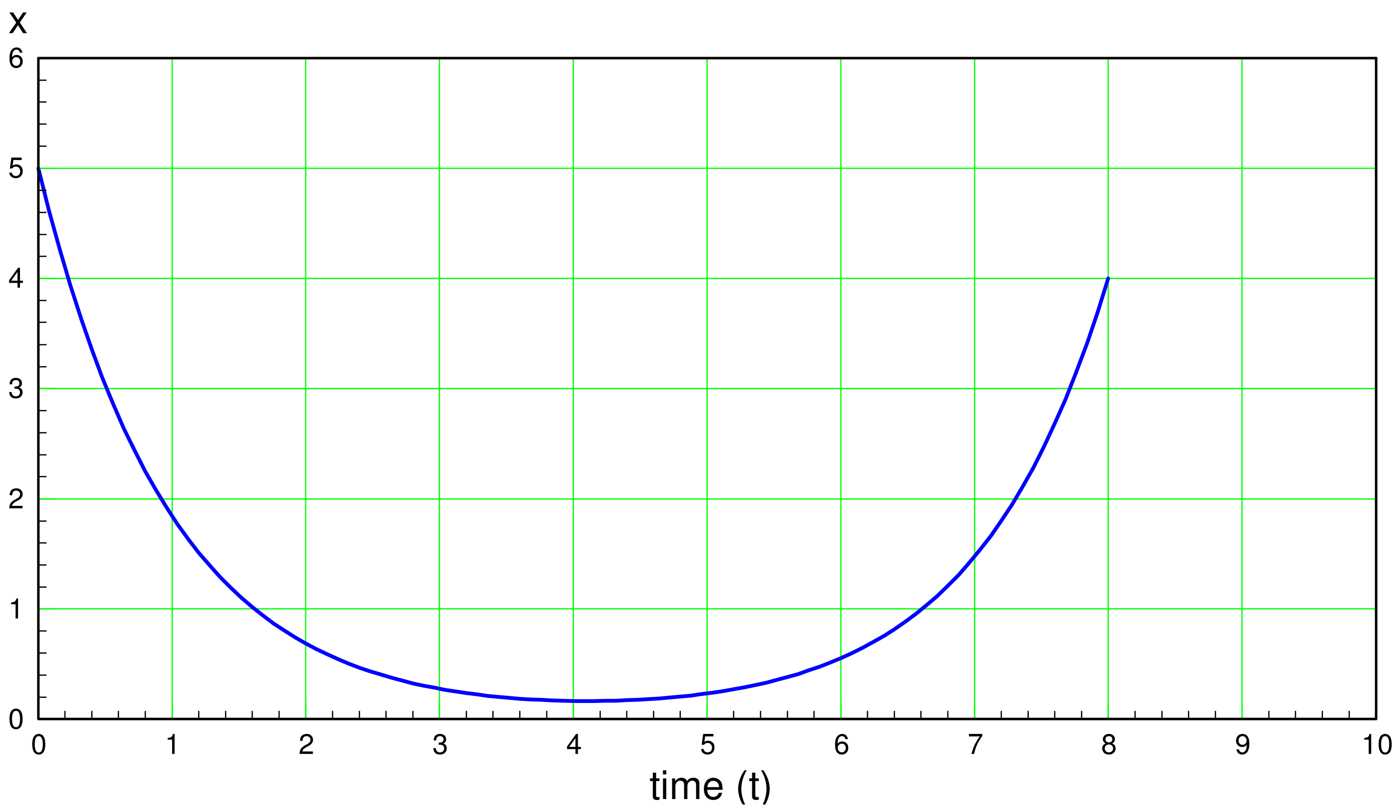
Plugging in the boundary conditions gives

$$x(0) = 5 = a + b$$

$$x(8) = 4 = 2981.0a + 0.0003b$$

gives

$$x(t) = 0.0013e^{t} + 4.9987e^{-t}$$



Optimal path of x(t) with the cost function $J = \int_0^1 (x^2 + \dot{x}^2)dt$

Example 2: 1st-Order Dynamic System

Find x(t) to minimize

$$J = \int_0^1 (x^2 + 9u^2)dt$$

subject to

- $\dot{x} = u$
- x(0) = 5
- x(8) = 4

Solution: Add a Legrange multiplier

$$F = x^2 + 9u^2 + m(\dot{x} - u)$$

You now have three sets of Euler LaGrange equations to solve:

$$F = x^2 + 9u^2 + m(\dot{x} - u)$$

i) With respect to x:

$$F_x - \tfrac{d}{dt}(F_{\dot{x}}) = 0$$

$$2x - \tfrac{d}{dt}(m) = 2x - \dot{m} = 0$$

ii) With respect to u:

$$F_u - \tfrac{d}{dt}(F_{\dot{u}}) = 0$$

$$18u - m = 0$$

iii) With respect to m:

$$F_m - \tfrac{d}{dt}(F_{\dot{m}}) = 0$$

$$\dot{x} - u = 0$$

Solving:

$$9\ddot{x} = x$$

$$(9s^2 - 1)x = 0$$

Either

- x = 0 (trivial solution), or
- s = { +1/3, -1/3 }

so

$$x(t) = ae^{t/3} + be^{-t/3}$$

Plugging in the constraints

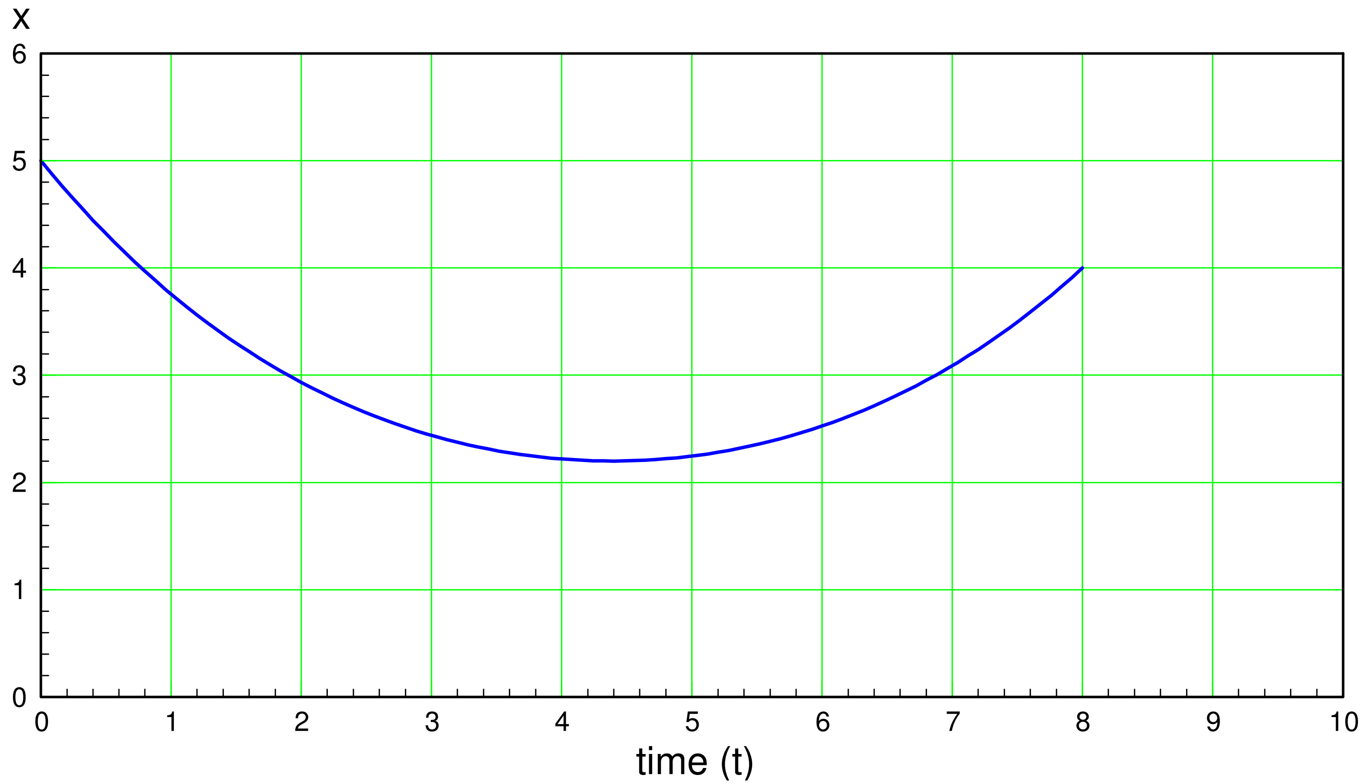
$$x(0) = 5 = a + b$$

$$x(8) = 4 = 14.3919a + 0.0695b$$

results in

$$x(t) = 0.2550e^{t/3} + 4.7450e^{-t/3}$$

$$u(t) = \dot{x}(t) = 0.0850e^{t/3} - 1.5817e^{-t/3}$$



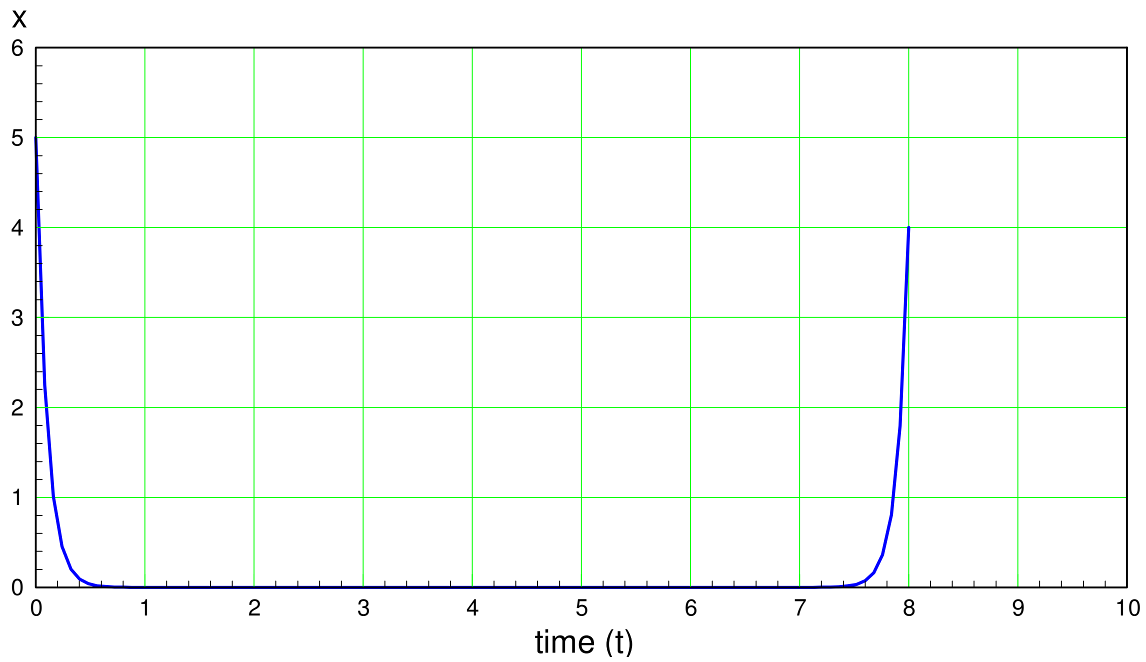
Note: Increase the weight on X pushes the curve closer to x = 0

If

$$J = \int_0^1 (100x^2 + u^2)dt$$

then

$$x = 7.219 \cdot 10^{-35}\ e^{10t} + 5e^{-10t}$$



Example 3: Find the functional to minimize

$$J = \int_a^b (X^T Q X + U^T R U) dt$$

subject to the constraint

$$\dot{X} = AX + BU$$

Solution: Add a LaGrange multiplier:

$$F = (X^T Q X + U^T R U) + 2M^T \left(AX + BU - \dot{X} \right)$$

The three Euler Legrange equations are then

$$2X^T Q + 2M^T A - \tfrac{d}{dt}(-2M^T) = 0$$

$$\dot{M} = -QX - A^T M$$

$$U = -R^{-1} B^T M$$

so you have the dyamic system

$$\begin{bmatrix} \dot{X} \\ \dot{M} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X \\ M \end{bmatrix}$$

which can be solved subject to the constraints on X(a) and X(b)

Full-State Feedback Formulation:

Assume

$$M = PX$$

so that the full-state feedback gains are

$$K = R^{-1} B^T P$$

Then the dynamics become

$$\dot{X} = (A - BR^{-1} B^T P) X$$

$$\dot{P} = -A^T P - PA - Q + PBR^{-1} B^T P$$

$$K = R^{-1} B^T P$$

If the feedback gains are constant, then

$$\dot{P} = 0$$

and

$$0 = -A^T P - PA - Q + PBR^{-1}B^T P \qquad \textit{algebraic Ricatti eqution}$$

$$K = -R^{-1}B^T P$$

Example: For the first-order system

$$\dot{x} = u$$

$$J = \int_0^\infty (qx^2 + ru^2)dt$$

m is

$$0 = -m^2 / r + q$$

or

$$m = \sqrt{qr}$$

$$k = \sqrt{q / r}$$

Note that

- Only the ratio of q/r matters
- As Q increases, the poles shift left (faster) as the square root of Q
- As R incrases, the poles shift right (slower) as the square root of R

Example: Find the optimal full-state feedback gain for

$$\dot{x} = -x + u$$

$$J = \int_0^\infty (x^2 + u^2) dt$$

$$q = 1$$

$$r = 1$$

The Ricatti equation becomes

$$0 = -A^T P - PA - Q + PBR^{-1} B^T P$$

$$0 = -2p - 1 + p^2$$

$$p = \{\, 0.4142,\ -2.4142 \,\}$$

$$k = \{\, 0.4142\ -2.4142 \,\}$$

This is a typical result.

- P (the Ricatti equation) is a quadratic equation - hence generally there are two solutions
- One of these solutions will be a minimum, the other a maximum. Since the feedback gain of -2.4142 results in an unstable system, that is the wrong solution (the maximum). Select the one that stabilizes the system.

The optimal feedback gain is

$$k = 0.4142$$

$$u = -kx$$