Cart and Pendulum & Gantry Dynamics

NDSU ECE 463/663

Lecture #7

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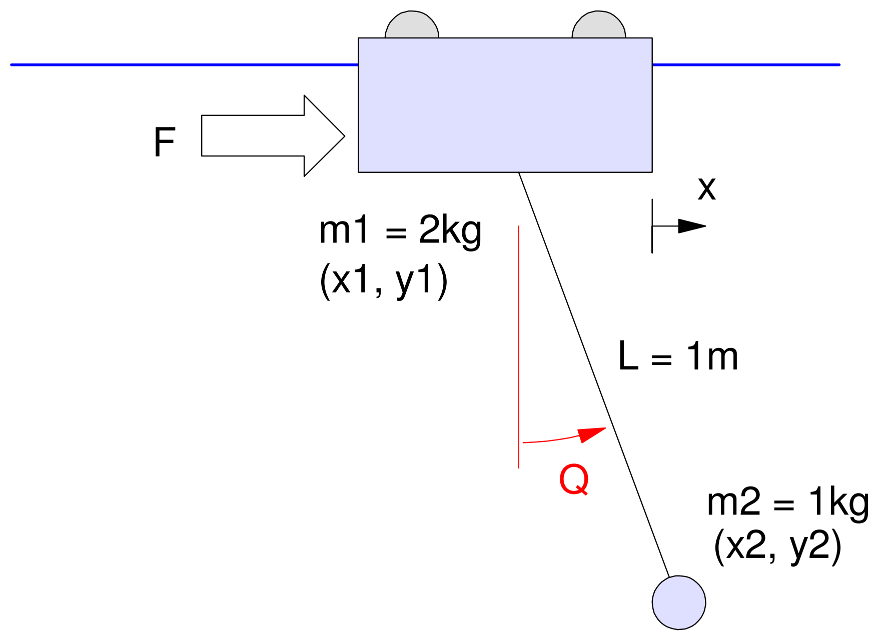
Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

Gantry System

- Overhead lift found in factories
- Pick up an engine block
- Move it somewhere else
- Tends to oscillate



Problem: Find the dynamics

- m1 = 2kg
- m1 = 1kg
- L = 1m

LaGrangian Formulation

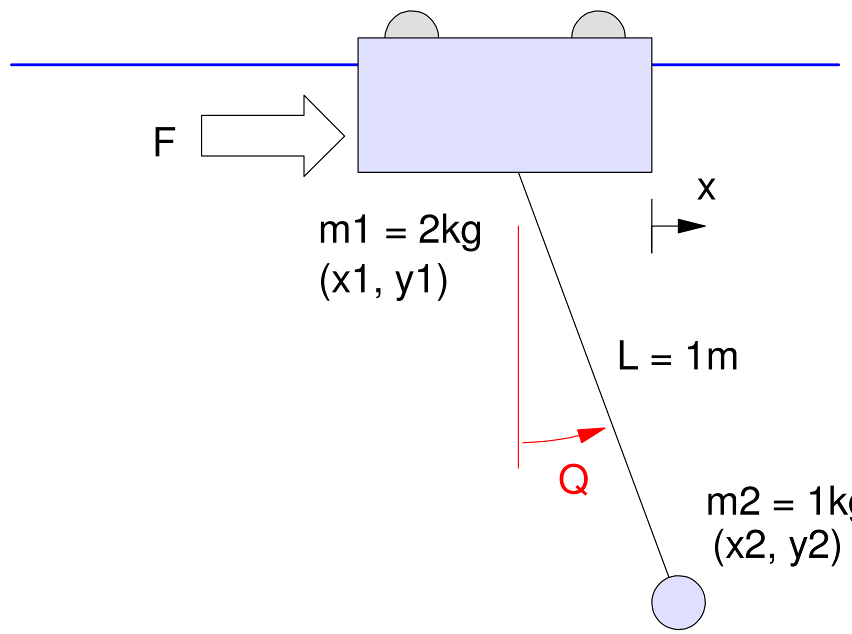
1) Define the energy in the system

Mass #1 (cart):

- position = (x_1, y_1)
- $x_1 = x$
- $y_1 = 0$

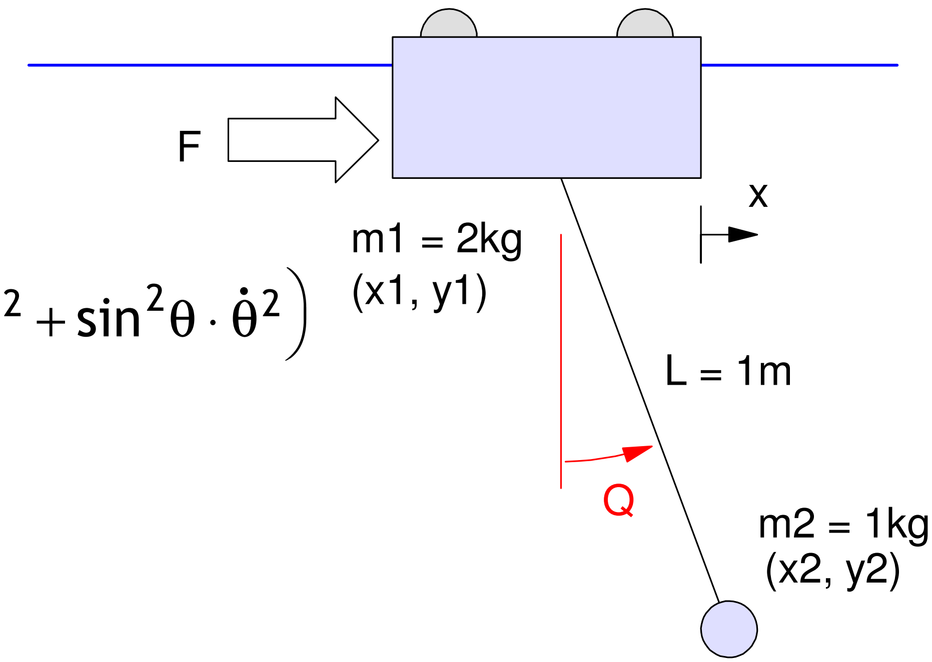
$$KE_1 = \frac{1}{2} m v^2 = \frac{1}{2} \cdot 2 \cdot \dot{x}^2$$

$$PE_1 = mgh = 0$$



Mass #2 (pendulum): position = (x_2, y_2)

- $x_2 = x_1 + \sin\theta$ $\qquad \dot{x}_2 = \dot{x}_1 + (\cos\theta)\dot{\theta}$
- $y_2 = -\cos\theta$ $\qquad \dot{y}_2 = (\sin\theta)\dot{\theta}$



$$KE_2 = \frac{1}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$KE_2 = \frac{1}{2}\left(\dot{x}_1^2 + 2\dot{x}_1\dot{\theta}_1\cos\theta + \cos^2\theta\cdot\dot{\theta}^2 + \sin^2\theta\cdot\dot{\theta}^2\right)$$

$$KE_2 = \frac{1}{2}\left(\dot{x}_1^2 + 2\dot{x}_1\dot{\theta}_1\cos\theta + \dot{\theta}^2\right)$$

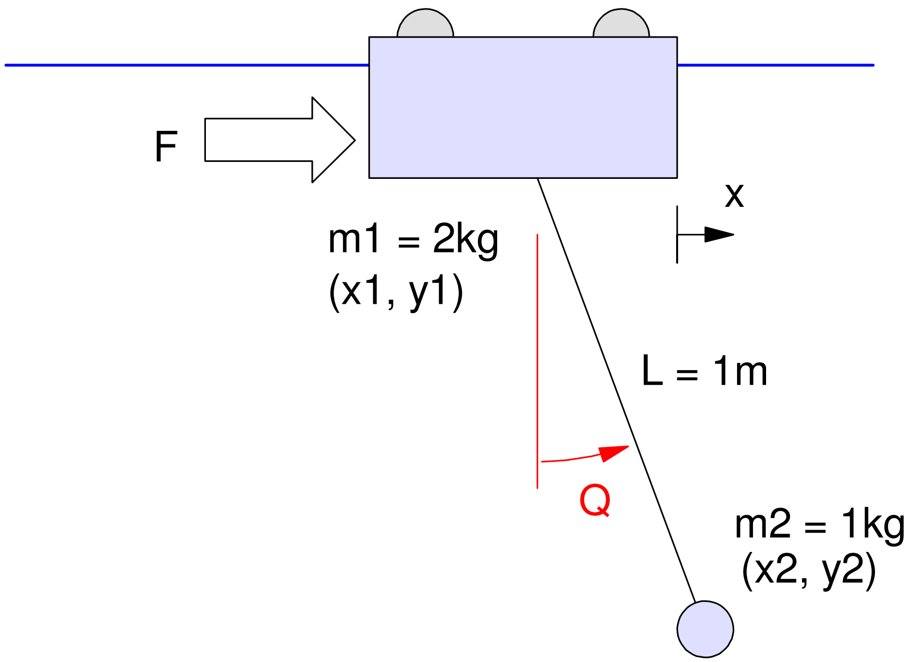
$$PE_2 = mgh = gy_2$$

$$PE_2 = -g\cos\theta$$

So, the Lagrangian is

$$L = \left(\dot{x}^2 + \tfrac{1}{2}\left(\dot{x}^2 + 2\dot{x}\dot{\theta}\cos\theta + \dot{\theta}^2\right)\right) - (-g\cos\theta)$$

$$L = \left(\tfrac{3}{2}\dot{x}^2 + \dot{x}\dot{\theta}\cos\theta + \tfrac{1}{2}\dot{\theta}^2\right) + g\cos\theta$$



The force on the base is

$$L = \left(\tfrac{3}{2}\dot{x}^2 + \dot{x}\dot{\theta}\cos\theta + \tfrac{1}{2}\dot{\theta}^2\right) + g\cos\theta$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \frac{d}{dt}\left(3\dot{x} + \dot{\theta}\cos\theta\right) - 0$$

$$F = 3\ddot{x} + \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta$$

The torque on the beam is

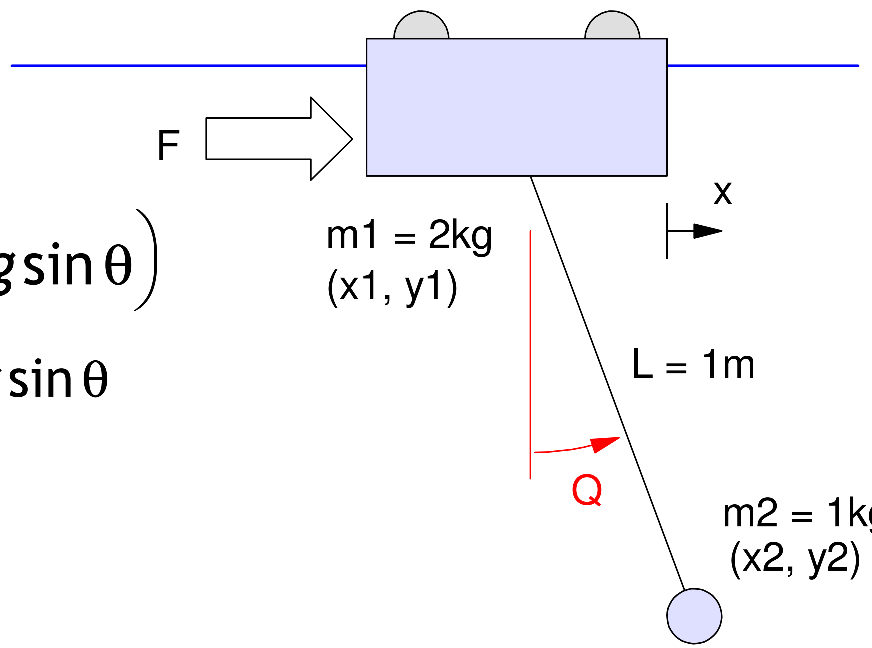
$$L = \left(\tfrac{3}{2}\dot{x}^2 + \dot{x}\dot{\theta}\cos\theta + \tfrac{1}{2}\dot{\theta}^2\right) + g\cos\theta$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \frac{d}{dt}\left(\dot{x}\cos\theta + \dot{\theta}\right) - \left(-\dot{x}\dot{\theta}\sin\theta - g\sin\theta\right)$$

$$T = \ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta + \ddot{\theta} + \dot{x}\dot{\theta}\sin\theta + g\sin\theta$$

$$\boxed{T = \ddot{x}\cos\theta + \ddot{\theta} + g\sin\theta}$$



So, the dynamics are

- This is what you use to simulate the system

$$\boxed{\begin{bmatrix} 3 & \cos\theta \\ \cos\theta & 1 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta}^2\sin\theta \\ -g\sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}F}$$

Linear Model

For small perturbations about 0,

$$\sin\theta \approx \theta \qquad \cos\theta \approx 1$$

$$\dot{\theta}^2 \approx 0 \qquad g = 9.8 \text{ m/s}^2$$

The dynamics linearized about zero are then

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ -9.8\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ 9.8\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F \right)$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 4.9\theta + 0.5F \\ -14.7\theta - 0.5F \end{bmatrix}$$

In State-Space form

$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 4.9 & 0 & 0 \\ 0 & -14.7 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix} F$$

Note that the eigenvalues of 'A' are

$$\{0, 0, j3.83, -j3.83\}$$

The pendulum swings back and forth at 3.83 rad/sec.

Eigenvalues and Eigenvectors

$$\lambda = \{0, 0, j3.83, -j3.38\}$$

$$\Lambda = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.316 \\ -0.950 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.316 \\ -0.950 \end{pmatrix} \right\}$$

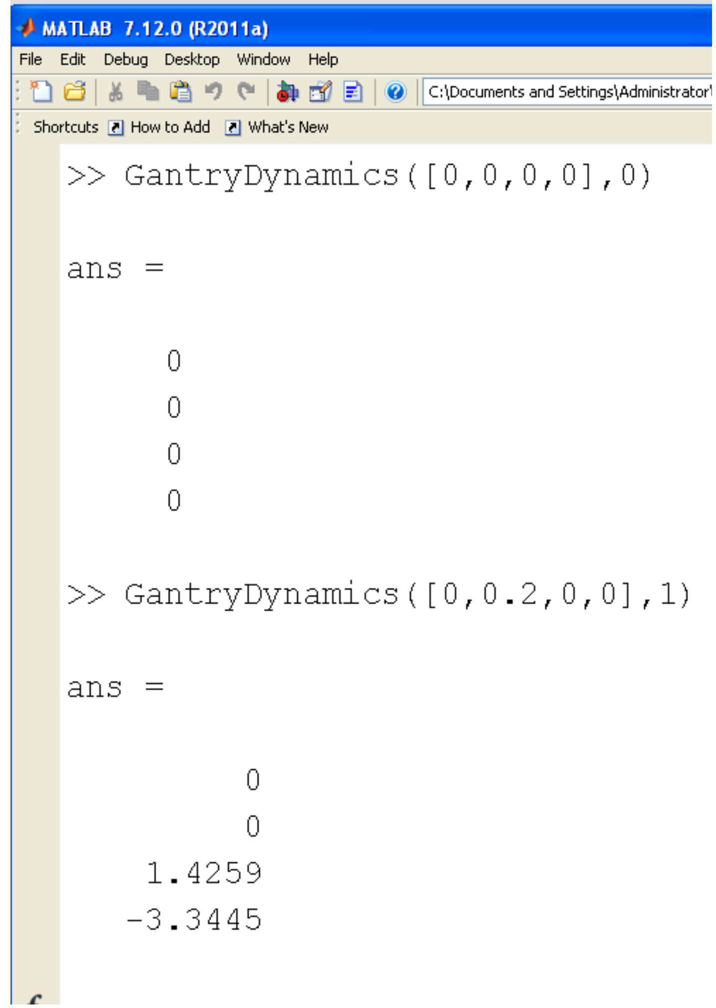
This means

- If you displace the cart by +1m, that state decays as e^{0t} (first eigenvector)
- If you start the cart moving right at 1m/s, it keeps drifting right decaying as e^{0t} (2nd eigenvector)
- If you displace the cart 0.316m right and swing the angle 0.950 rad left, the cart oscillates back and forth at 3.83 rad/sec (3rd eigenvector)
- If yo start out at (0, 0) but make the initia velocity 0.316 m/s right and the initial angular velocity -0.950 rad/sec, the cart osicllates back and forth at 3.83 rad/sec (4th eigenvector)

Nonlinear Dynamics with Matlab

$$\begin{bmatrix} 3 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta}^2 \sin\theta \\ -g\sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

```
function [ dX ] = GantryDynamics( X, F )
%cart dynamics
% cart = 2kg
% ball = 1kg
% length = 1m
% X = [x, q, dx, dq]
x = X(1);
q = X(2);
dx = X(3);
dq = X(4);
g = 9.8;
M = [3, cos(q); cos(q), 1];
A = [dq*dq*sin(q); -g*sin(q)];
B = [1;0];
d2X = inv(M) * (A + B*F);
dX = [dx; dq; d2X];
end
```

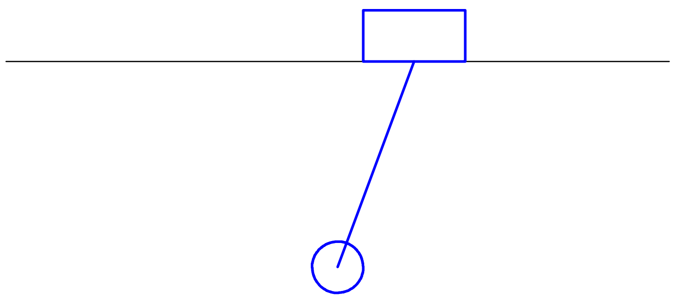


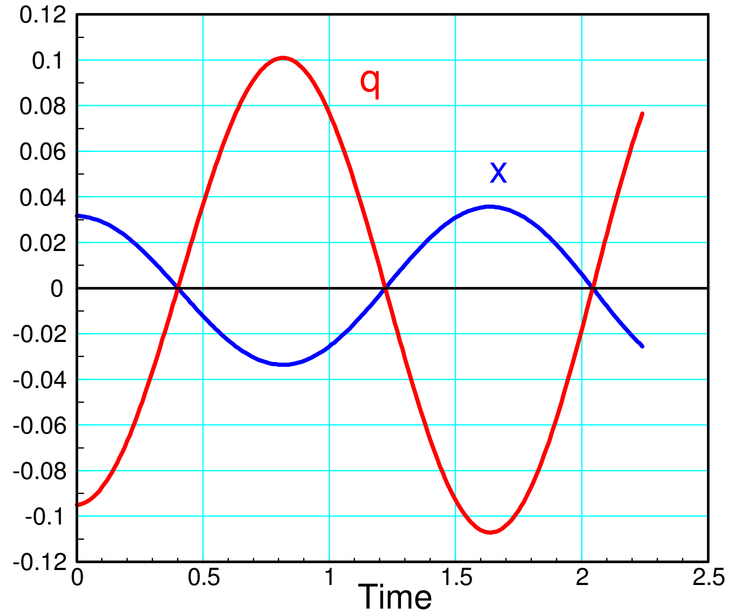
Animation and Eigenvectors

```
X = [0.0316;-0.0950;0;0];
dX = zeros(4,1);
Ref = 0;
dt = 0.01;
U = 0;
t = 0;
y = [];

while(t < 10)
 dX = GantryDynamics(X, U);
 X = X + dX * dt;
 t = t + dt;
 GantryDisplay(X, Ref);
 y = [y ; X(1), X(2) ];
 end

hold off
t = [0:length(y)-1]' * dt;
plot(t,y);
```





PD Control of a Gantry System

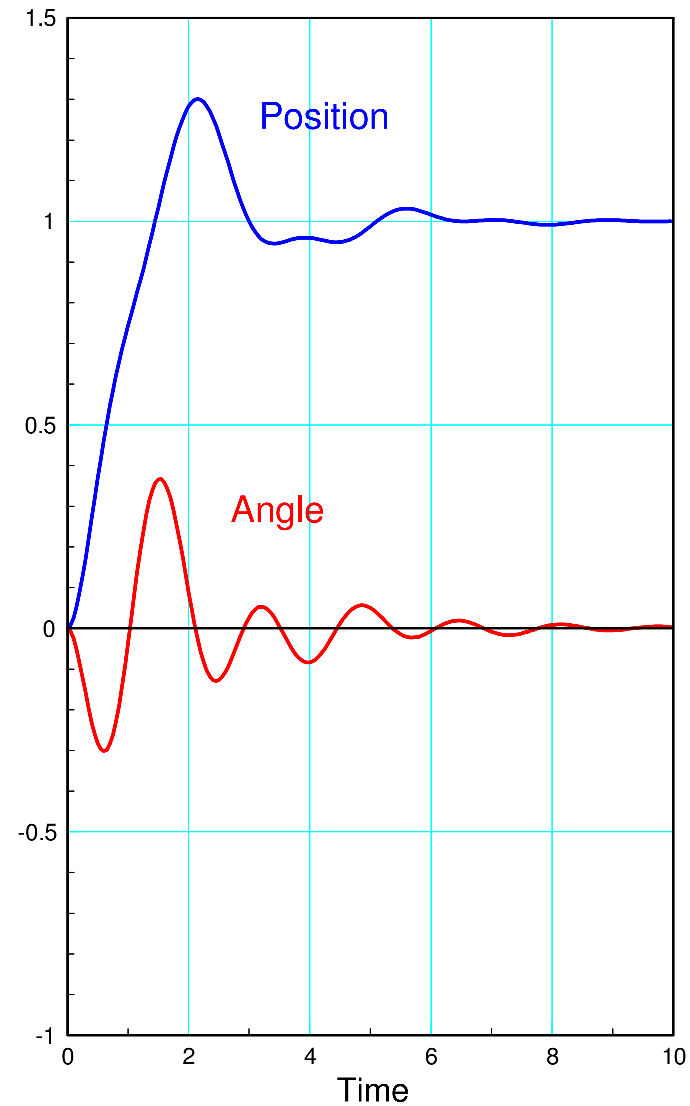
$$F = P(R - x) + D\left(\dot{R} - \dot{x}\right)$$

Trial and error to find P and D

- ECE 461 gives other methods

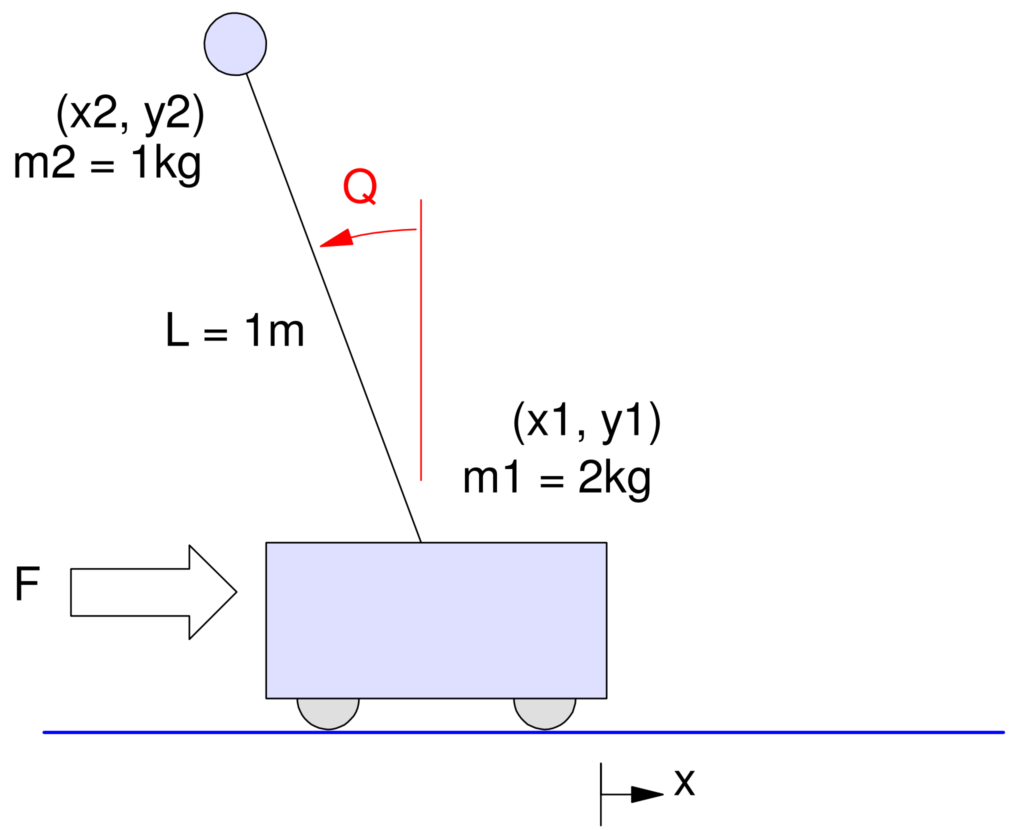
```
X = [0;0;0;0];
dX = zeros(4,1);
Ref = 1;
dt = 0.01;
U = 0;
t = 0;
y = [];
P = 10;
D = 5;

while(t < 10)
   U = P*(Ref - X(1)) + D*(0 - X(3));
   dX = GantryDynamics(X, U);
   X = X + dX * dt;
   t = t + dt;
   GantryDisplay(X, Ref);
   y = [y ; X(1), X(2) ];
   end
```



Cart & Pendulum (Cart.m)

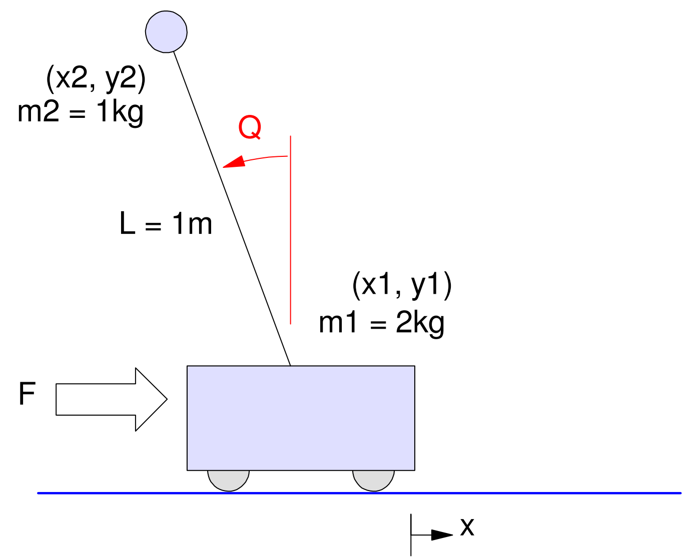
- Balance a rocket as you move it to the launch pad
- Balance a yard stick on your hand



Same dynamics as before, just change the direction of gravity

$g = -9.8\ m/s^2$:

$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4.9 & 0 & 0 \\ 0 & 14.7 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ x' \\ \theta' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix}F$$



Eigenvalues

$\{0, 0, 3.83, -3.83\}$

As expected, an inverted pendulum is unstable.

Eigenvalues and Eigenvectors

$$\lambda = \{0, 0, +3.83, -3.38\}$$

$$\Lambda = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.0798 \\ 0.2394 \\ -0.3060 \\ 0.9180 \end{pmatrix}, \begin{pmatrix} 0.0798 \\ -0.2394 \\ -0.3060 \\ 0.9180 \end{pmatrix} \right\} \text{change}$$

This means

- In theory, you can get the pendulum to balance if you get the initial conidtion just right (the 4th eigenvector)
- In practice, any non-zero initial condition on the 3rd eigenvector and the pendulum falls over

Nonlinear Dynamics in Matlab

$$\begin{bmatrix} 3 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta}^2 \sin\theta \\ +g\sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

Only change: Sign of gravity term

- Results in positive feedback (unstable)

```
function [ dX ] = CartDynamics( X, F )
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% length = 1m
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q = X(2);
dx = X(3);
dq = X(4);
g = 9.8;
M = [3, cos(q); cos(q), 1];
A = [dq*dq*sin(q); g*sin(q)];
B = [1;0];
d2X = inv(M) * (A + B*F);
dX = [dx; dq; d2X];
end
```
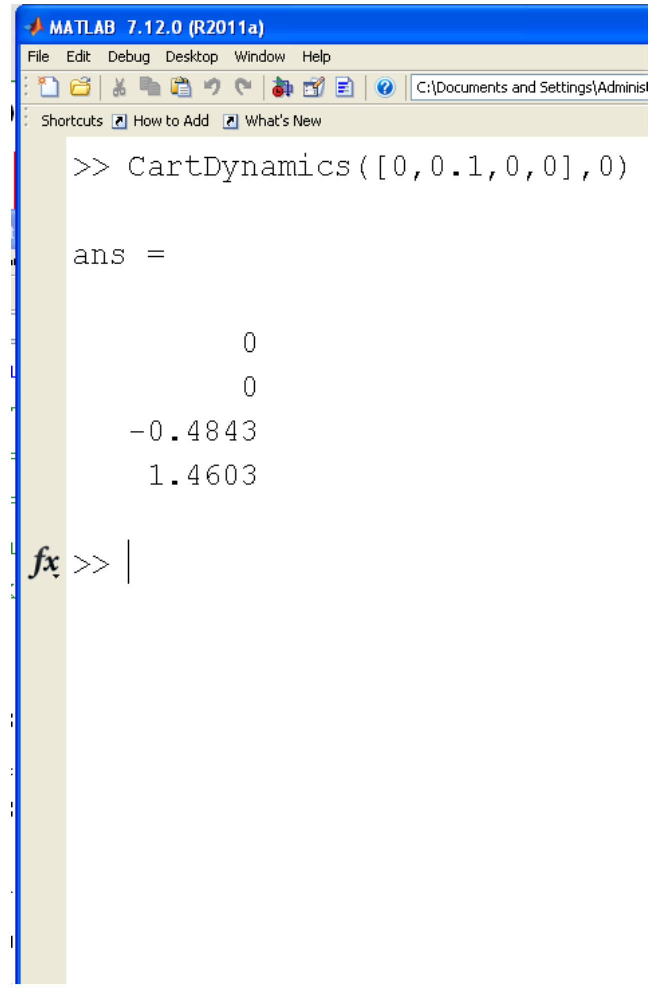



Open-Loop Response

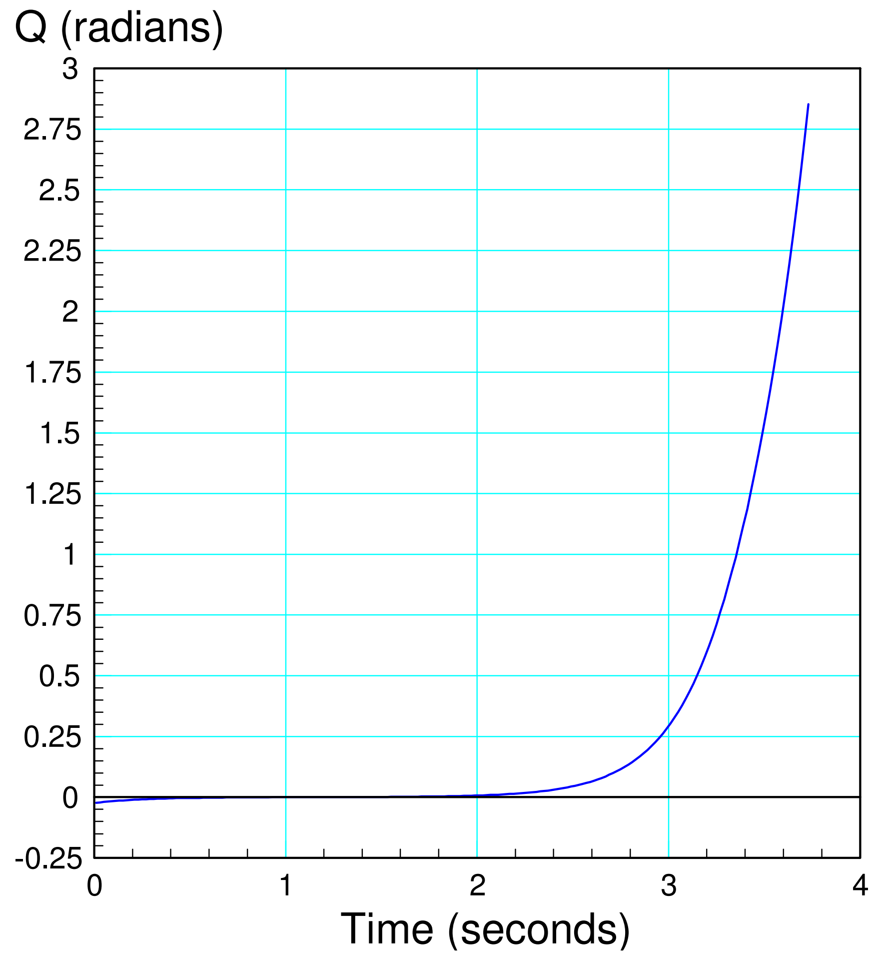
- U = 0

The system is unstable

Any initial condition results in the pendulum falling

PD Control also doesn't work

- $U = P(R - x) + D\left(\dot{R} - \dot{x}\right)$
- No set of PD gains wills stabilize this sytem



Full-State Feedback

It *is* possible to stabilize this sytem

- One solution uses all four states (full-state feedback)
- Upcoming topic for ECE 463