Ball and Beam System

NDSU ECE 463/663

Lecture #8

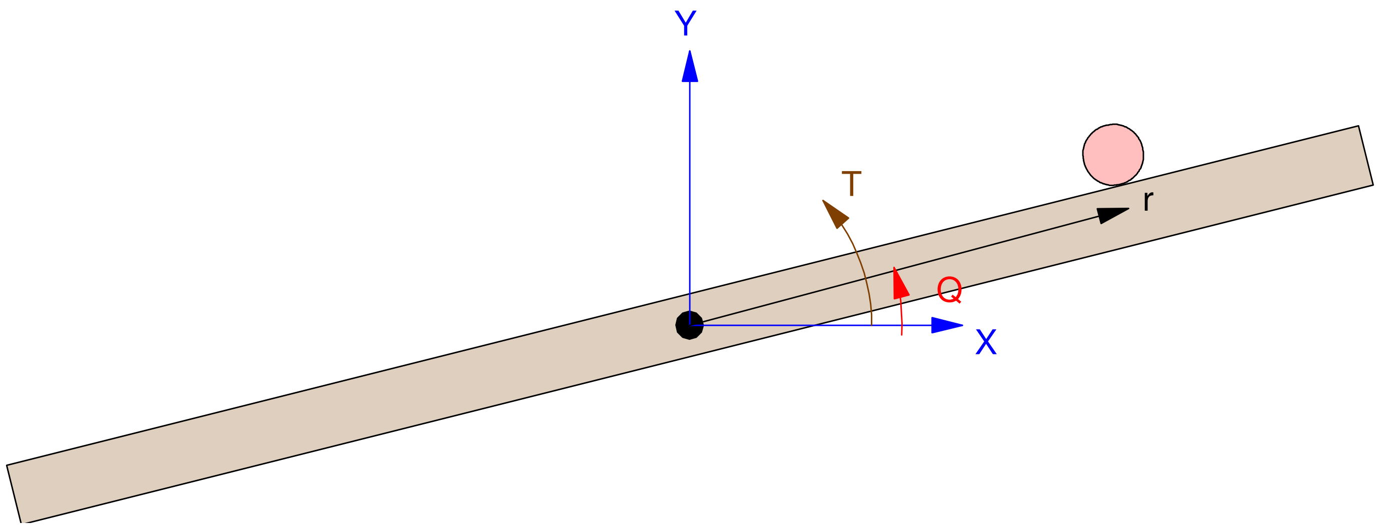
Inst: Jake Glower

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Ball and Beam System

A ball rolls along a beam of length L.

- The ball has a mass of 1kg
- The beam has a rotational inertia of 0.2 kg m^2
- A motor applies a torque to the beam (T).
- The goal is to balance the ball at a certain spot (1.0 meters in this case).



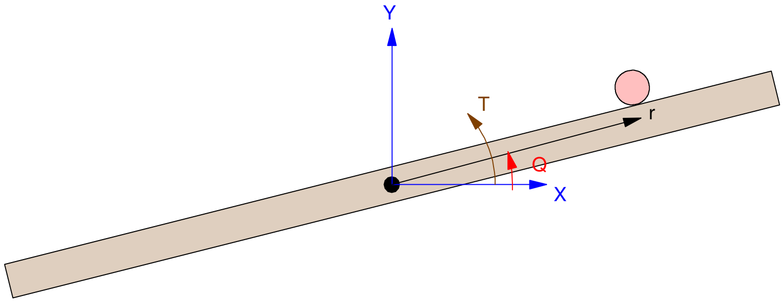
LaGrangian Dynamics

1) Write the position and velocity of the ball

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \qquad \dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$$

2) Write the energy in the system



$$PE = mgy = mgr\sin\theta$$

$$KE = \tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{5}m\dot{r}^2$$

Note that the kinetic energy has three terms:

- The rotational energy of the beam
- The translational energy of the ball, and
- The rotational energy of the ball. Assume a solid sphere.

Substituting...

$$KE = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{5}m\dot{r}^2$$

$$KE = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\left(\left(\dot{r}\cos\theta - r\sin\theta\dot{\theta}\right)^2 + \left(\dot{r}\sin\theta + r\cos\theta\dot{\theta}\right)^2\right) + \frac{1}{5}m\dot{r}^2$$

$$KE = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{5}m\dot{r}^2$$

Pluggin in numbers (J = 0.2, m = 1)

$$KE = 0.1\dot{\theta}^2 + 0.7\dot{r}^2 + 0.5r^2\dot{\theta}^2$$

So, the LaGrangian is

$$L = KE - PE$$

$$L = \left(0.1\dot{\theta}^2 + 0.7\dot{r}^2 + 0.5r^2\dot{\theta}^2\right) - (gr\sin\theta)$$

Force on the Ball

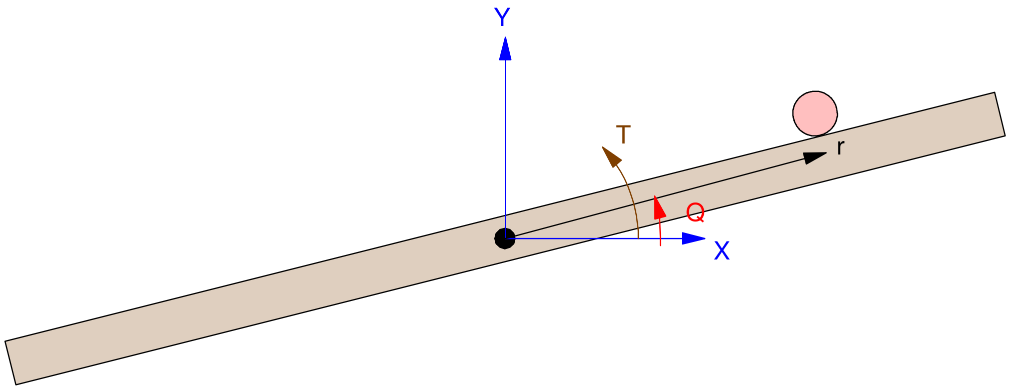
- i.e. if there was a motor driving the ball or friction

$$L = \left(0.1\dot{\theta}^2 + 0.7\dot{r}^2 + 0.5r^2\dot{\theta}^2\right) - (gr\sin\theta)$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right)$$

$$F = \frac{d}{dt}(1.4\dot{r}) - \left(r\dot{\theta}^2 - g\sin\theta\right)$$

$$\boxed{F = 1.4\ddot{r} - r\dot{\theta}^2 + g\sin\theta}$$



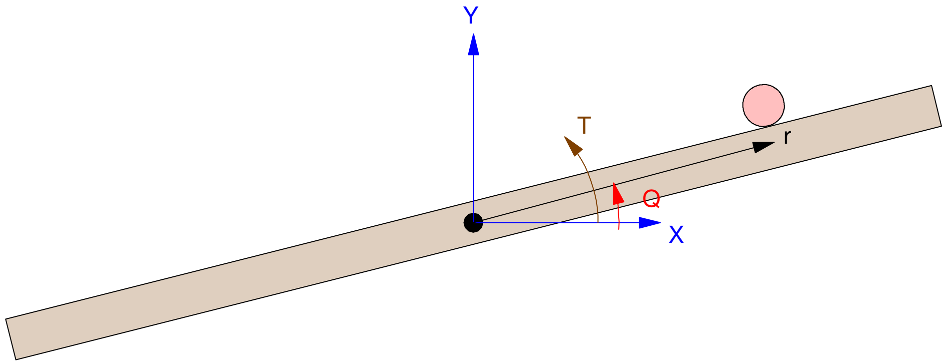
Torque on the Beam

$$L = \left(0.1\dot{\theta}^2 + 0.7\dot{r}^2 + 0.5r^2\dot{\theta}^2\right) - (gr\sin\theta)$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \frac{d}{dt}\left(0.2\dot{\theta} + r^2\dot{\theta}\right) - (-gr\cos\theta)$$

$$\boxed{T = 0.2\ddot{\theta} + r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\cos\theta}$$



Putting it together:

$$\begin{bmatrix} 1.4 & 0 \\ 0 & 0.2 + r^2 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} r\dot{\theta}^2 - g\sin\theta \\ -2r\dot{r}\dot{\theta} - gr\cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

Nonlinear Model

$$\begin{bmatrix} 1.4 & 0 \\ 0 & 0.2+r^2 \end{bmatrix} \begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} r\dot{\theta}^2 - g\sin\theta \\ -2r\dot{r}\dot{\theta} - gr\cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

Linearized Model:

- r = 1.0, angle = zero, F = 0, m = 1kg, J = 0.2kg m^2

$$\begin{bmatrix} 1.4 & 0 \\ 0 & 1.2 \end{bmatrix} \begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -g\theta \\ -gr \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

In state-space

$$s\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -7 & 0 & 0 \\ -8.167 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.833 \end{bmatrix} T$$

Eigenvalues and Eigenvectors are:

$$\lambda = \left\{ -2.7497 \;\; j2.7397 \;\; -j2.7497 \;\; +2.7497 \right\}$$

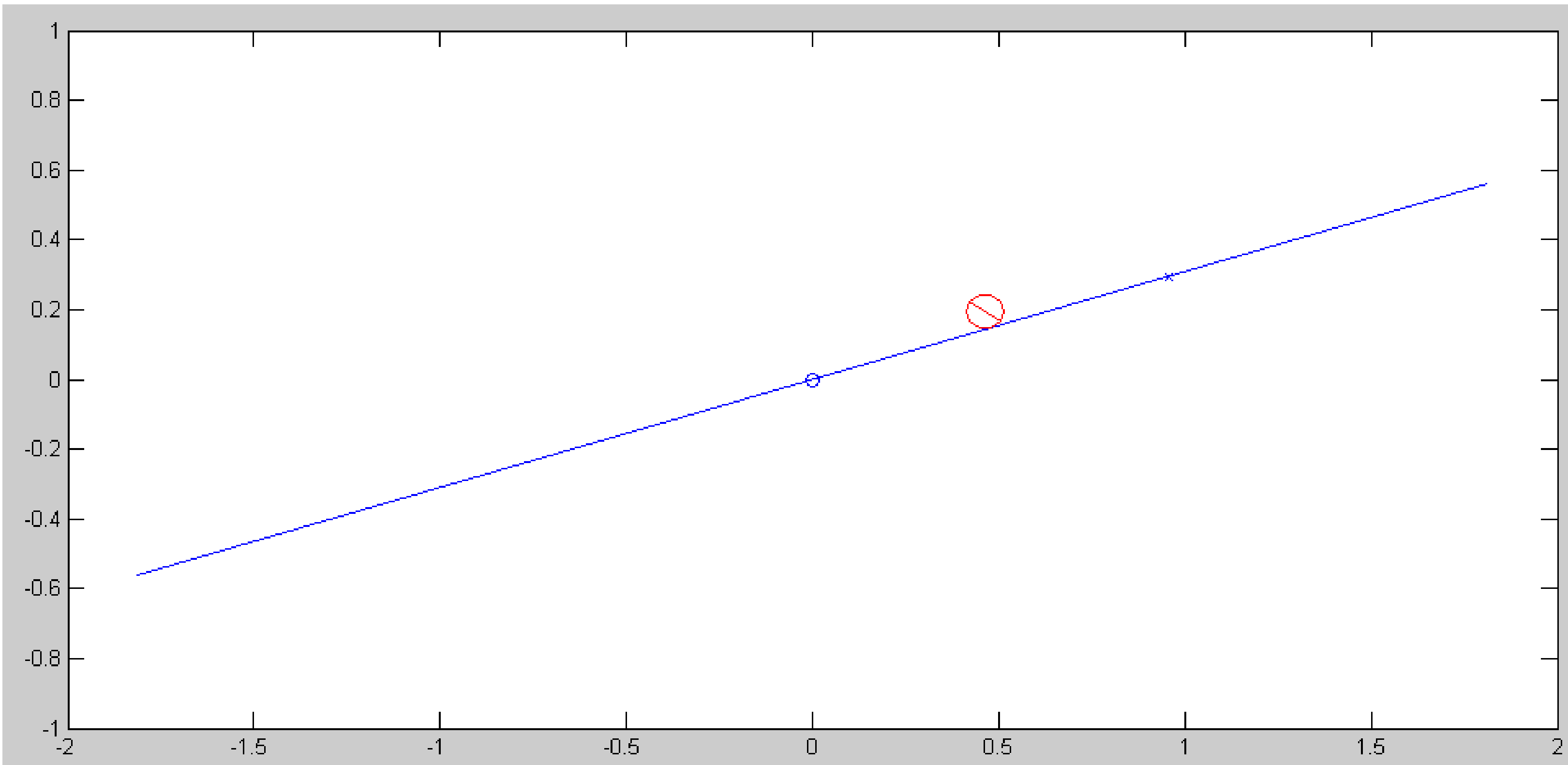
$$\Lambda = \left\{ \begin{pmatrix} -0.2322 \\ 0.2508 \\ 0.6384 \\ -0.68896 \end{pmatrix} \begin{pmatrix} 0.2322 \\ 0.2508 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -0.6384 \\ -0.6896 \end{pmatrix} \begin{pmatrix} -0.2322 \\ 0.2508 \\ -0.6384 \\ 0.6896 \end{pmatrix} \right\}$$

This is very difficult to control by hand

- Open loop unstable, plus
- Two poles on the jw axis

Matlab Animation Files:

```
                     r    q    dr  dq     Ref
>> BeamDisplay([0.5,0.3, 0,   0]', 1)
```



BeamDynamics

$$\begin{bmatrix}1.4 & 0\\ 0 & 0.2+r^2\end{bmatrix}\begin{bmatrix}\ddot{r}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} r\dot{\theta}^2 - g\sin\theta \\ -2r\dot{r}\dot{\theta} - gr\cos\theta\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}T$$

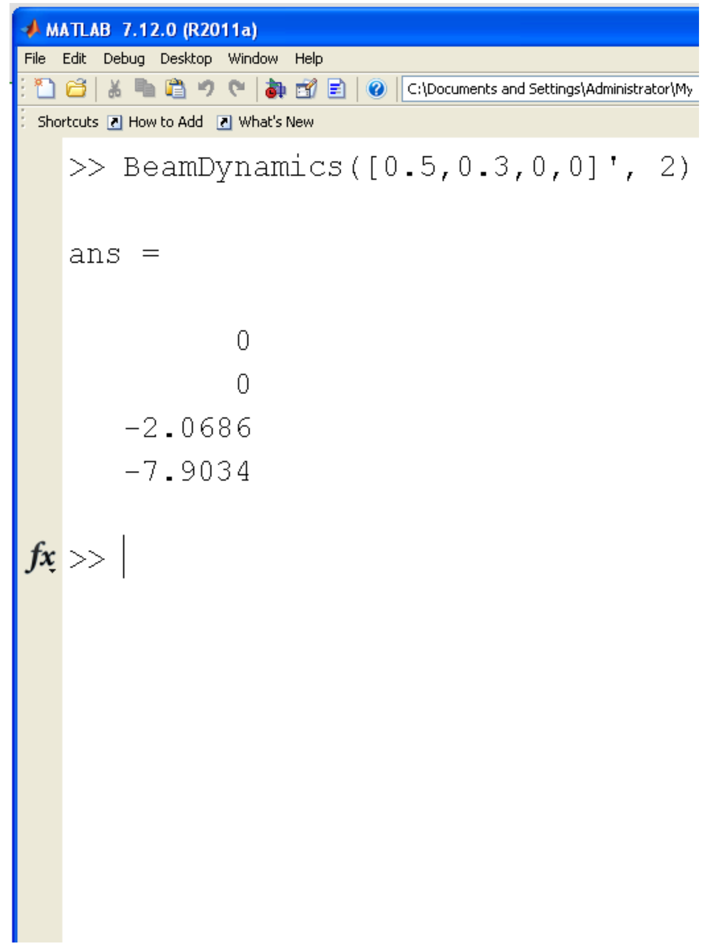
```
function [dX] = BeamDynamics( X, T )
% Ball and Beam:

r = X(1);
q = X(2);
dr = X(3);
dq = X(4);
g = 9.8;

M = [1.4, 0; 0, 0.2 + r*r];
B1 = r*dq*dq - g*sin(q);
B2 = T - 2*r*dr*dq - g*r*cos(q);

ddX = inv(M)*[B1; B2];
dX = [dr; dq; ddX];

end
```

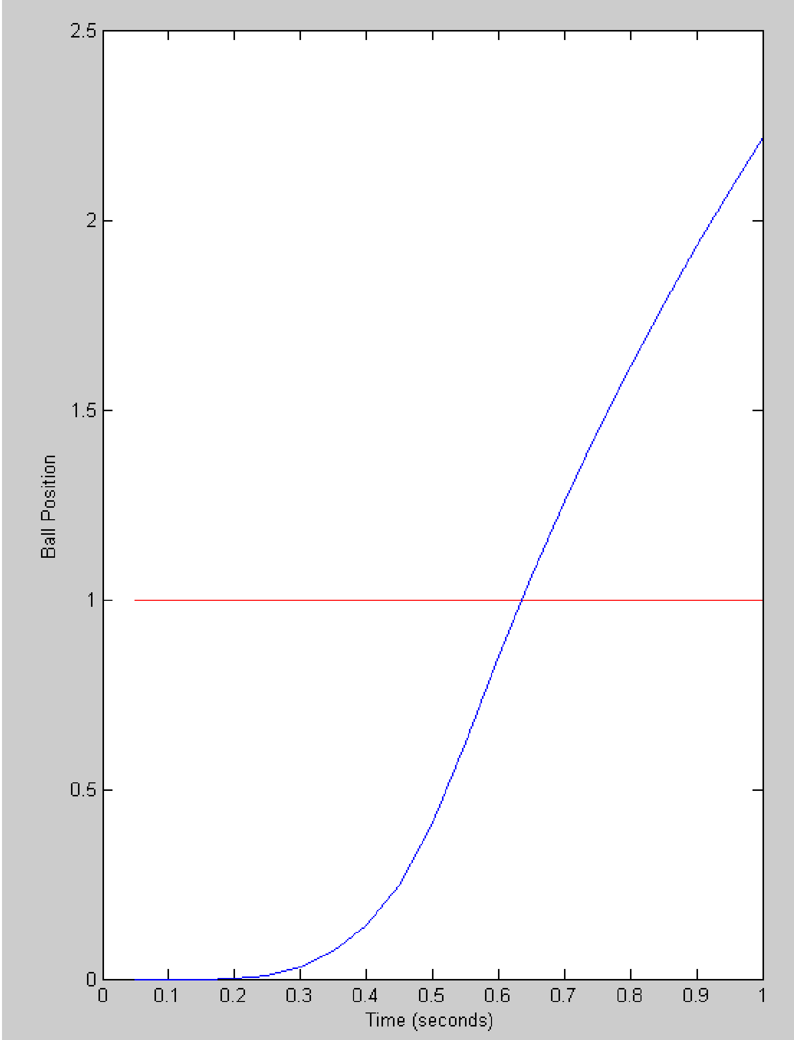


Beam:

- PD Control won't stabilize the system
- (ECE 461 Method)

```
% Ball & Beam System
%     [x   q   dx   dq]
X = [0, 0, 0, 0]';
dt = 0.01;
t = 0;
Ref = 1;
y = [];

while(t < 5)
 U = 10*(Ref-X(1)) + 5*(0-X(3));
 dX = BeamDynamics(X, U);
 X = X + dX * dt;
 y = [y ; Ref, X(1)];
 t = t + dt;
 BeamDisplay(X, Ref);
 end
```



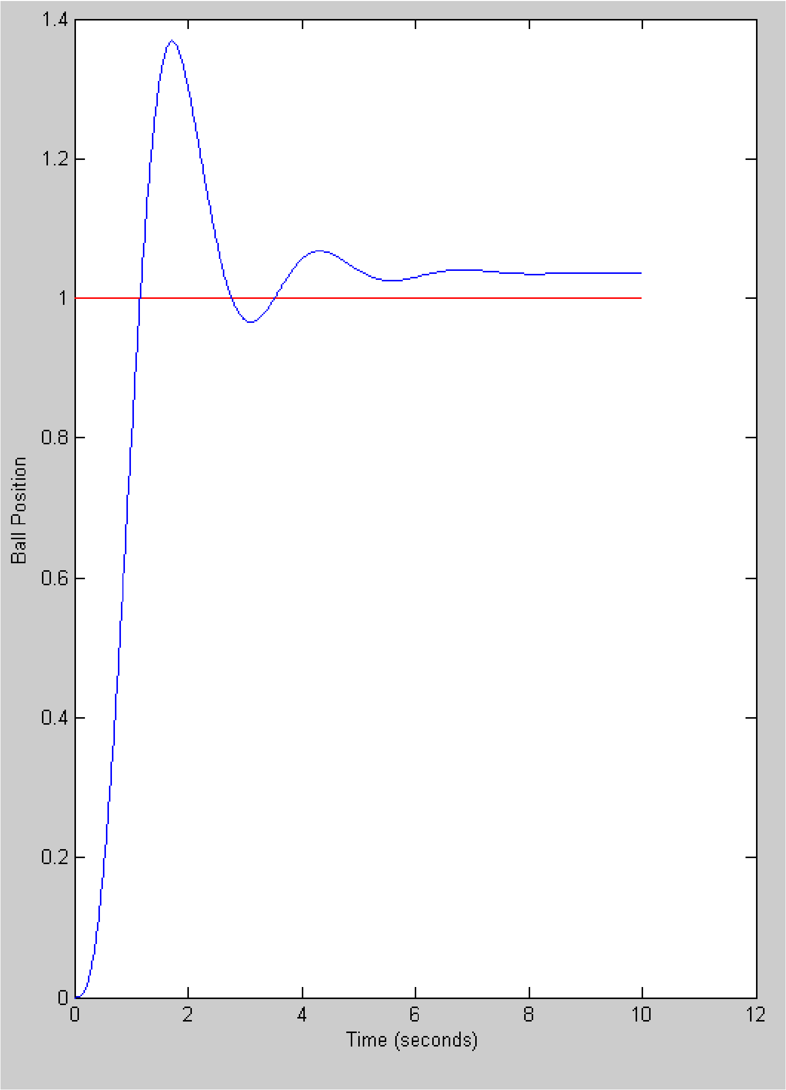
Full-State can stabilize this system

- ECE 463 method

```
% Ball & Beam System

X = [0, 0, 0, 0]';
dt = 0.01;
t = 0;
Kx = [-59   93  -32   23 ];
Kr = -51;
Ref = 1;
y = [];

while(t < 10)
 U = Kr*Ref - Kx*X;
 dX = BeamDynamics(X, U);
 X = X + dX * dt;
 y = [y ; Ref, X(1)];
 t = t + dt;
 BeamDisplay(X, Ref);
 end
```



Variation: Input = $\dot{\theta}$

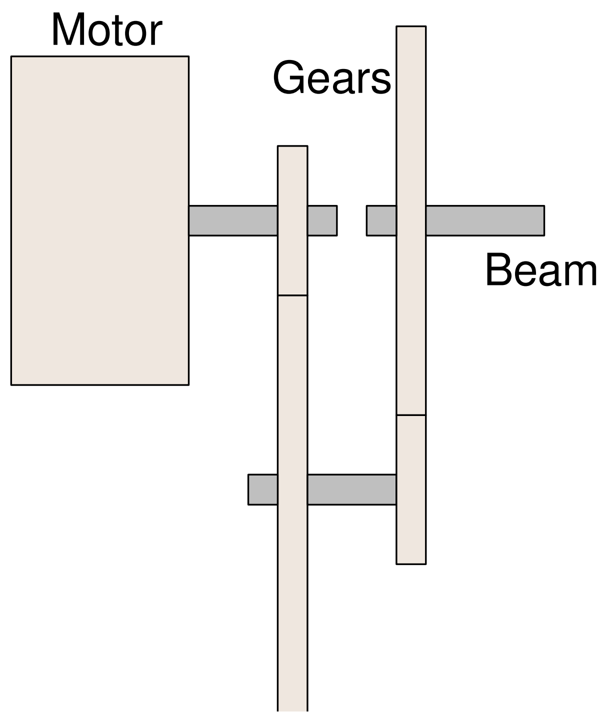
- Geared motor driving the beam
- Voltage ≈ Speed

Nonlinear Dynamics become

$$1.4\ddot{r} = r\dot{\theta}^2 - g\sin\theta$$

The Linearized Dynamics become

$$s\begin{bmatrix} r \\ \dot{r} \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} r \\ \dot{r} \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\dot{\theta}$$



Poles are at {0, 0, 0} (tripple integrator}

- Still very hard to control

Variation

Add friction

$$F = -\dot{r}$$

Add feedback to the motor

$$\dot{\theta} = 10(R - \theta)$$

The nonlinear dynamics become

$$1.4\ddot{r} = r\dot{\theta}^2 - g\sin\theta + F$$

The linearized dyanmics become

$$s\begin{bmatrix} r \\ \dot{r} \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.7 & 7 \\ 0 & 0 & -10 \end{bmatrix}\begin{bmatrix} r \\ \dot{r} \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} R$$

The poles are now

$$\lambda = \left\{ 0,\ -0.7,\ -10 \right\}$$

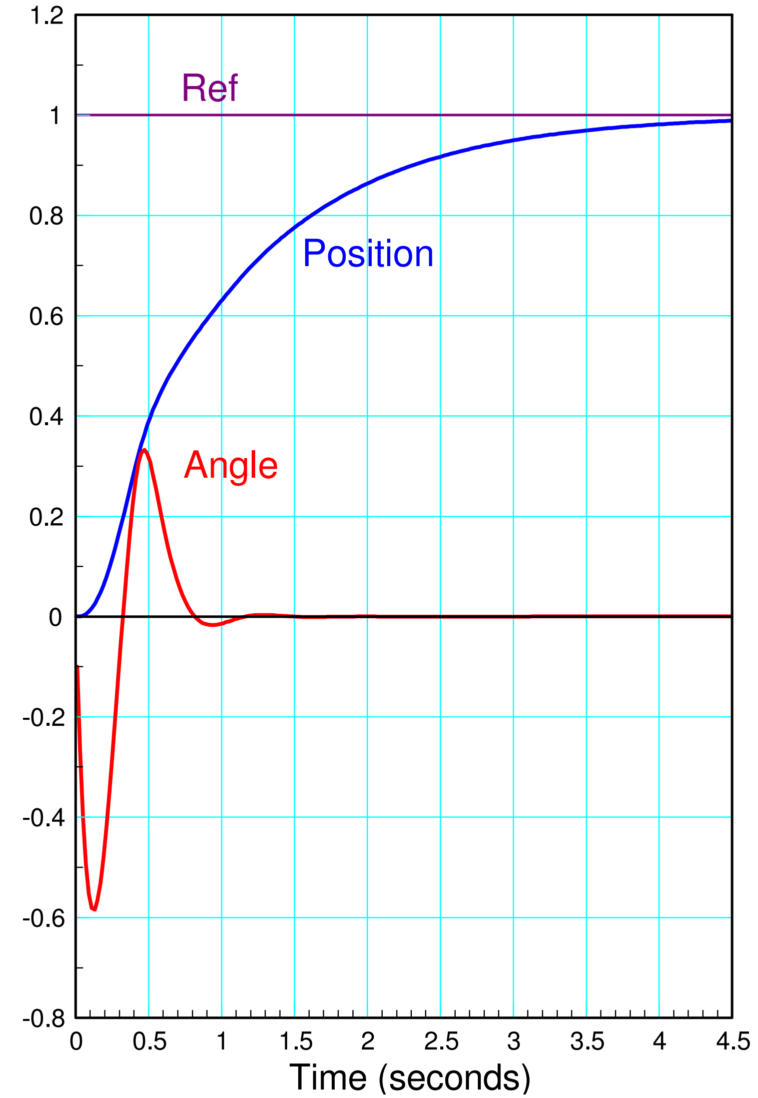
PD Control now works

- PD Control (ECE 461) works best with systems are open-loop stable
- Full-State feedback (ECE 463) works with any system

```
while(t < 5)
 U = 1*(Ref - r) + 1*(0 - dr);
 dq = 10*(-U - q);
 ddr = (r*dq*dq - 9.8*sin(q) - dr);

 dr = dr + ddr*dt;
 r = r + dr*dt;
 q = q + dq*dt;
 t = t + dt;

 BeamDisplay([r; q; dr; dq], Ref);
 end
```



Summary

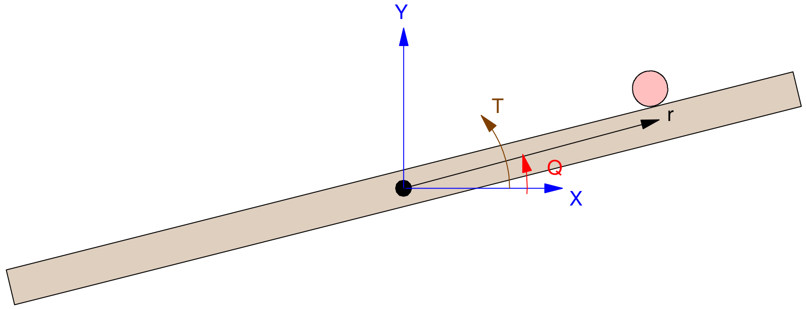
- J = 0.2 kg m^2, M = 1.0kg

results in

$$\begin{bmatrix} 1.4 & 0 \\ 0 & 0.2+r^2 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} r\dot{\theta}^2 - g\sin\theta \\ -2r\dot{r}\dot{\theta} - gr\cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

At r = 1.0m

$$s\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -7 & 0 & 0 \\ -8.167 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.833 \end{bmatrix} T$$



We'll be designing feedback controllers for this system in the upcoming lectures