Dynamics of a 2-Link Arm

NDSU ECE 463/663

Lecture #9

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Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Robotic Arms

https://blog.technavio.com/wp-content/uploads/2018/07

Industrial robots typically have 3 degrees of freedom

- base - shoulder - elbow
- Allows you to reach any spot in 3-space

Plus 3 degrees of freedom at the tip

- Wrist
- Allows any orientation at the tip

LaGrange formulation of dynamics works

- About a 10-page derivation

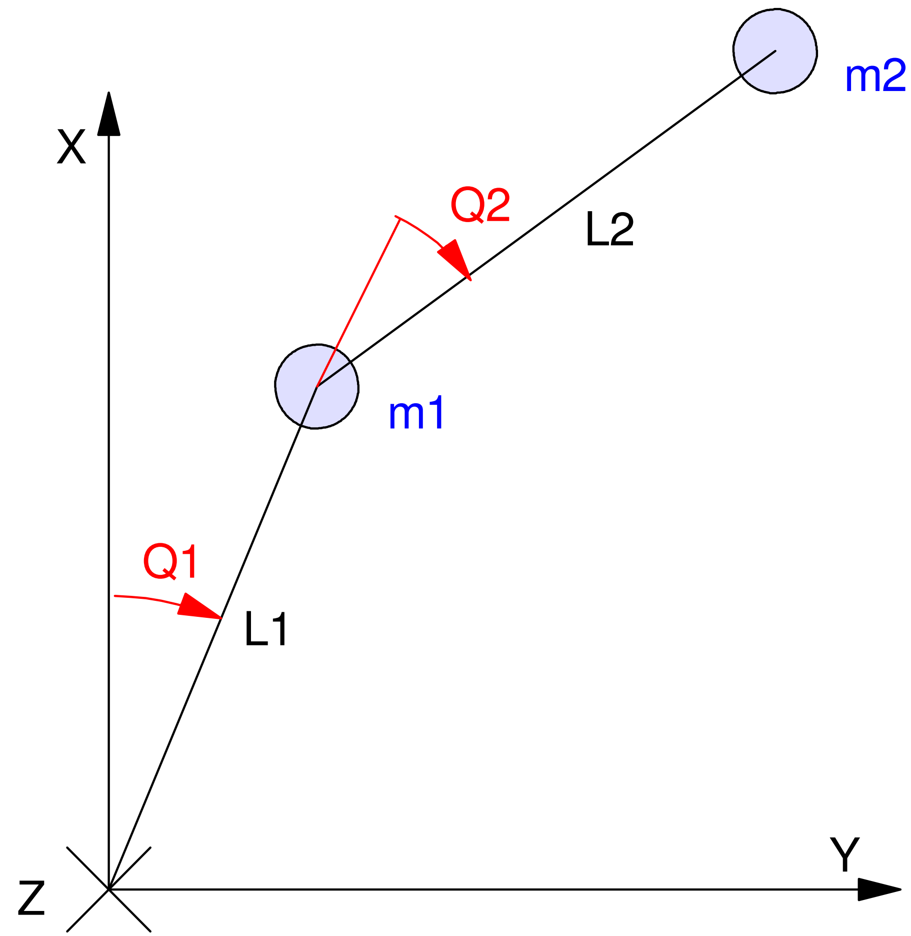
2-Link Arm

- Ignore the wrist
- Fix the base at zero degrees

Results in a 2-link robotic arm

- The robot can move about the XY plane
- Zero position points straight up

Tip = f(Q1, Q2)



2-Link Dynamics:

Determine the dynamics

- L1 = L2 = 1m
- m1 = 1kg
- m2 = 1kg

Step 1: Determine the (x,y) position and velocity of each mass

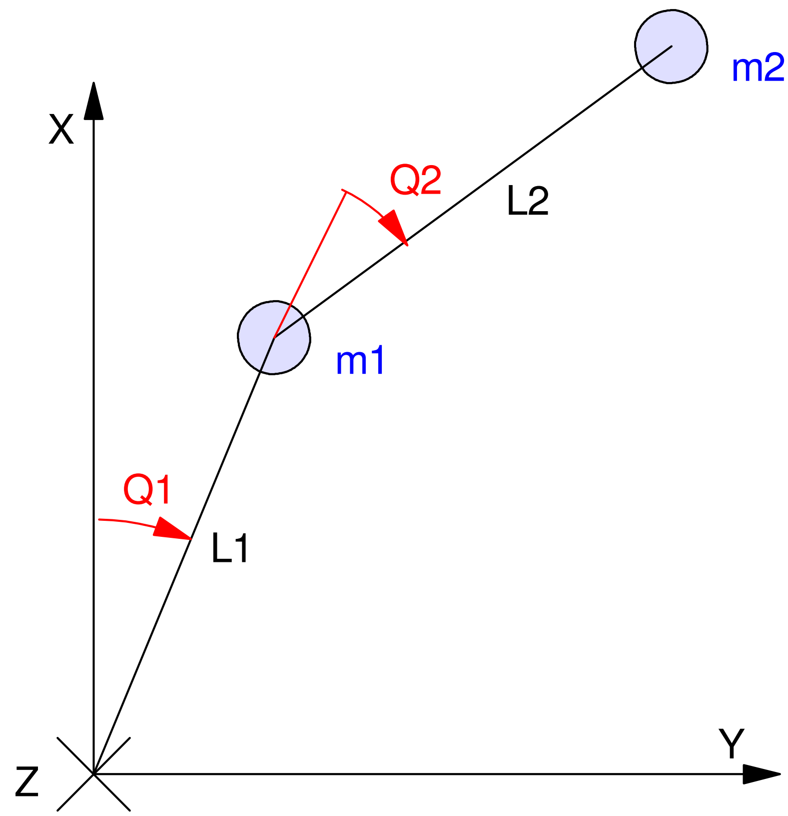
mass 1:

$$x_1 = \cos\theta_1$$

$$\dot{x}_1 = -\sin(\theta_1)\dot{\theta}_1$$

$$y_1 = \sin(\theta_1)$$

$$\dot{y}_1 = \cos(\theta_1)\dot{\theta}_1$$



mass 2:

$$x_2 = x_1 + \cos(\theta_1 + \theta_2)$$

$$\dot{x}_2 = -\sin(\theta_1)\dot{\theta}_1 - \sin(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$y_2 = y_1 + \sin(\theta_1 + \theta_2)$$

$$\dot{y}_2 = \cos(\theta_1)\dot{\theta}_1 + \cos(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

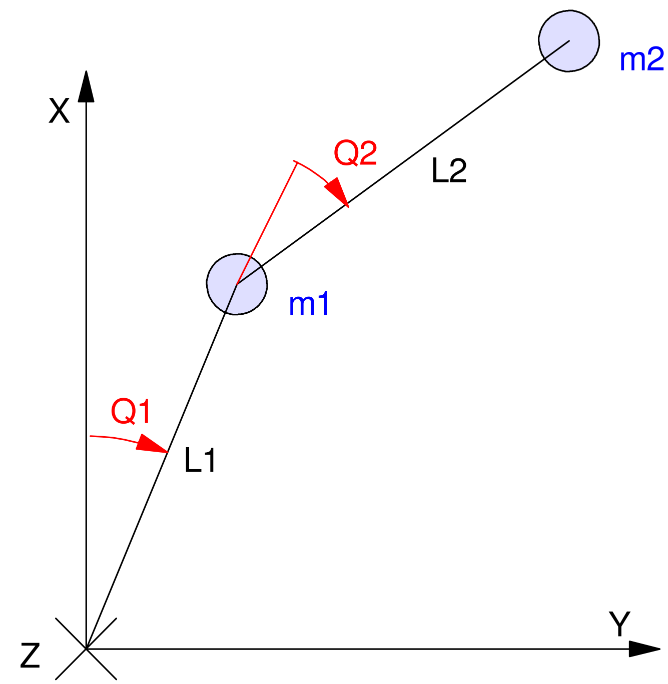
Using shorthand notation

$$x_1 = c_1 \qquad \dot{x}_1 = -s_1\dot{\theta}_1$$

$$y_1 = s_1 \qquad \dot{y}_1 = c_1\dot{\theta}_1$$

$$x_2 = c_1 + c_{12} \qquad \dot{x}_2 = -s_1\dot{\theta}_1 - s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$y_2 = s_1 + s_{12} \qquad \dot{y}_2 = c_1\dot{\theta}_1 + c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

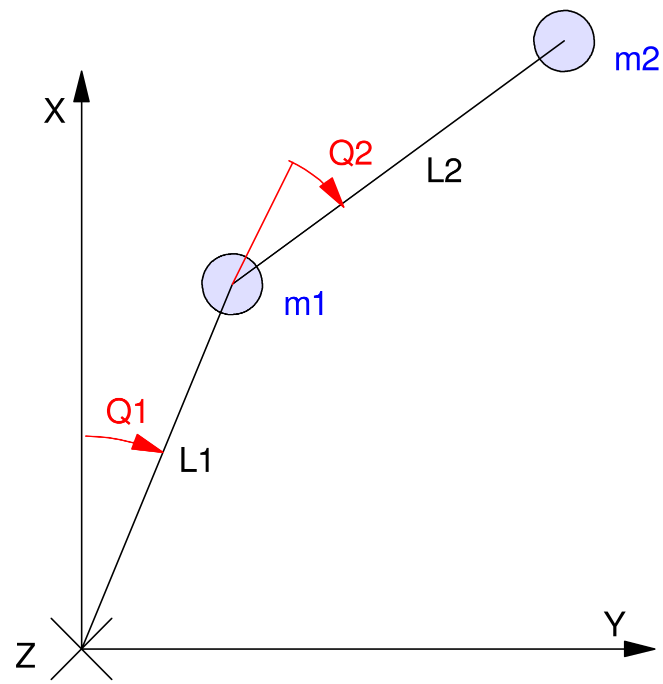


Step 2: Form the kinetic and potential energy:

Mass 1: $m_1 = 1$

$$KE = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2 \right)$$

$$KE = \frac{1}{2} \left(\left(-s_1 \dot{\theta}_1 \right)^2 + \left(c_1 \dot{\theta}_1 \right)^2 \right)$$



Note that $\sin^2 + \cos^2 = 1$

$$KE = \frac{1}{2} \dot{\theta}_1^2$$

$$PE = m_1 g x_1$$

$$PE = g c_1$$

Mass 2: $m_2 = 1$

$$KE = \tfrac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2 \right)$$

$$KE = \tfrac{1}{2} \left(\left(-s_1 \dot{\theta}_1 - s_{12} \left(\dot{\theta}_1 + \dot{\theta}_2 \right) \right)^2 + \left(c_1 \dot{\theta}_1 + c_{12} \left(\dot{\theta}_1 + \dot{\theta}_2 \right) \right)^2 \right)$$

$$KE = \tfrac{1}{2} \left(s_1^2 \dot{\theta}_1^2 + s_{12}^2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2 + c_1^2 \dot{\theta}_1^2 + c_{12}^2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2 \right)$$

$$+ s_1 s_{12} \dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2 \right) + c_1 c_{12} \dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2 \right)$$

$$KE = \tfrac{1}{2} \left(\dot{\theta}_1^2 + \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2 \right) + c_1 c_{12} \dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2 \right) + s_1 s_{12} \dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2 \right)$$

From trigonometry

$$\sin(a)\sin(b)+\cos(a)\cos(b)=\cos(a-b)$$

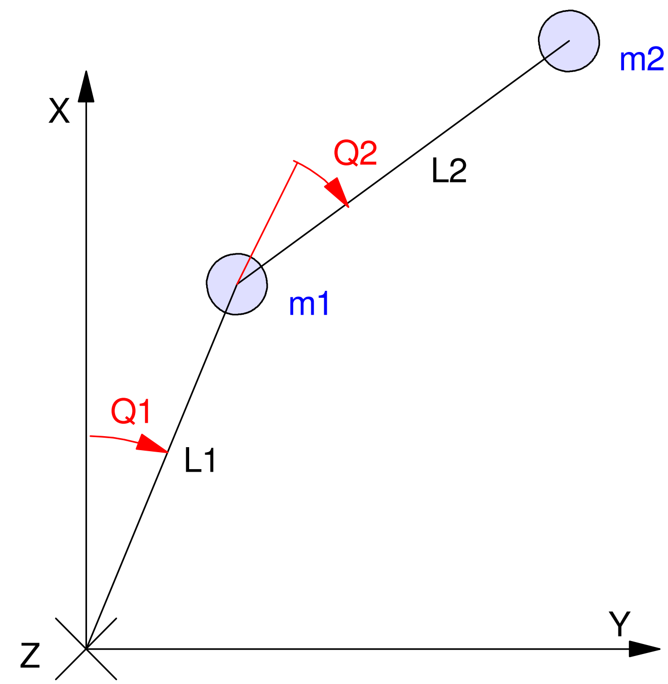
resulting in

$$KE=\frac{1}{2}\left(\dot{\theta}_1^2+\left(\dot{\theta}_1+\dot{\theta}_2\right)^2\right)+c_2\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)$$

$$KE=(1+c_2)\dot{\theta}_1^2+\tfrac{1}{2}\dot{\theta}_2^2+(1+c_2)\dot{\theta}_1\dot{\theta}_2$$

$$PE=m_2 g x_2$$

$$PE=g(c_1+c_{12})$$



Step 3: Form the LaGrangian:

$$L = KE - PE$$

$$L = \left(\left(\tfrac{1}{2}\dot{\theta}_1^2\right) + \left((1 + c_2)\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + (1 + c_2)\dot{\theta}_1\dot{\theta}_2\right)\right) - ((gc_1) + (g(c_1 + c_{12})))$$

$$L = (1.5 + c_2)\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + (1 + c_2)\dot{\theta}_1\dot{\theta}_2 - 2gc_1 - gc_{12}$$

Step 4: Take the partials

$$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right)$$

$$T_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right)$$

Starting with θ_1:

$$L = (1.5 + c_2)\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + (1 + c_2)\dot{\theta}_1\dot{\theta}_2 - 2gc_1 - gc_{12}$$

$$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right)$$

$$T_1 = \frac{d}{dt}\Big((3 + 2c_2)\dot{\theta}_1 + (1 + c_2)\dot{\theta}_2\Big) - (-2gs_1 - gs_{12})$$

$$T_1 = \Big((3 + 2c_2)\ddot{\theta}_1 - 2s_2\dot{\theta}_1\dot{\theta}_2 + (1 + c_2)\ddot{\theta}_2 - $$

Moving on to θ_2

$$L = (1.5 + c_2)\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + (1 + c_2)\dot{\theta}_1\dot{\theta}_2 - 2gc_1 - gc_{12}$$

$$T_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right)$$

$$T_2 = \frac{d}{dt}\left(\dot{\theta}_2 + (1 + c_2)\dot{\theta}_1\right) - \left(-s_2\dot{\theta}_1^2 - s_2\dot{\theta}_1\dot{\theta}_2 - gs_{12}\right)$$

$$T_2 = \left(\ddot{\theta}_2 + (1 + c_2)\ddot{\theta}_1 - s_2\dot{\theta}_1\dot{\theta}_2\right) + \left(s_2\dot{\theta}_1^2 + s_2\dot{\theta}_1\dot{\theta}_2 + gs_{12}\right)$$

$$T_2 = \ddot{\theta}_2 + (1 + c_2)\ddot{\theta}_1 + s_2\dot{\theta}_1^2 + gs_{12}$$

Net result:

$$T_1 = \left((3+2c_2)\ddot{\theta}_1 - 2s_2\dot{\theta}_1\dot{\theta}_2 + (1+c_2)\ddot{\theta}_2 - s_2\dot{\theta}_2^2 \right) + (2gs_1 + gs_{12})$$

$$T_2 = \ddot{\theta}_2 + (1+c_2)\ddot{\theta}_1 + s_2\dot{\theta}_1^2 + gs_{12}$$

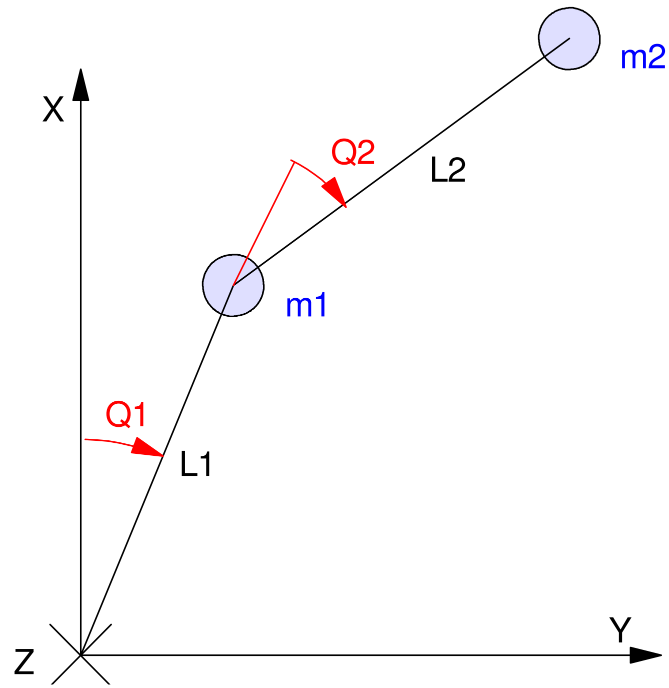
To solve, rewrite this in matrix form:

$$\begin{bmatrix} (3+2c_2) & (1+c_2) \\ (1+c_2) & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + g\begin{bmatrix} 2s_1 + s_{12} \\ s_{12} \end{bmatrix}$$

Mass Matrix * Acceleration = Torque + Coriolis Forces + Gravity

Given the joint angles, velocities, gravity, and input torques, you can compute the joint accelerations as

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} (3+2c_2) & (1+c_2) \\ (1+c_2) & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + g \begin{bmatrix} 2s_1 + s_{12} \\ s_{12} \end{bmatrix} \right)$$



MatLab Code:

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} (3+2c_2) & (1+c_2) \\ (1+c_2) & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + g\begin{bmatrix} 1s_1 + s_{12} \\ s_{12} \end{bmatrix} \right)$$

```
function [ ddQ ] = TwoLinkDynamics( Q, dQ, T )

q1 = Q(1);
q2 = Q(2);
dq1 = dQ(1);
dq2 = dQ(2);
c1 = cos(q1);
s1 = sin(q1);
s2 = sin(q2);
c2 = cos(q2);
s12 = sin(q1+q2);
c12 = cos(q1+q2);
g = 9.8;
M = [ 3 + 2*c2 , 1+c2 ; 1+c2 , 1];
C = [ 2*s2*dq1*dq2 + s2*dq2*dq2 ; -s2*dq1*dq1 ];
G = g*[2*s1 + s12 ; s12 ];
ddQ = inv(M) * ( T + C + G );
end
```

2-Link Arm in Freefall

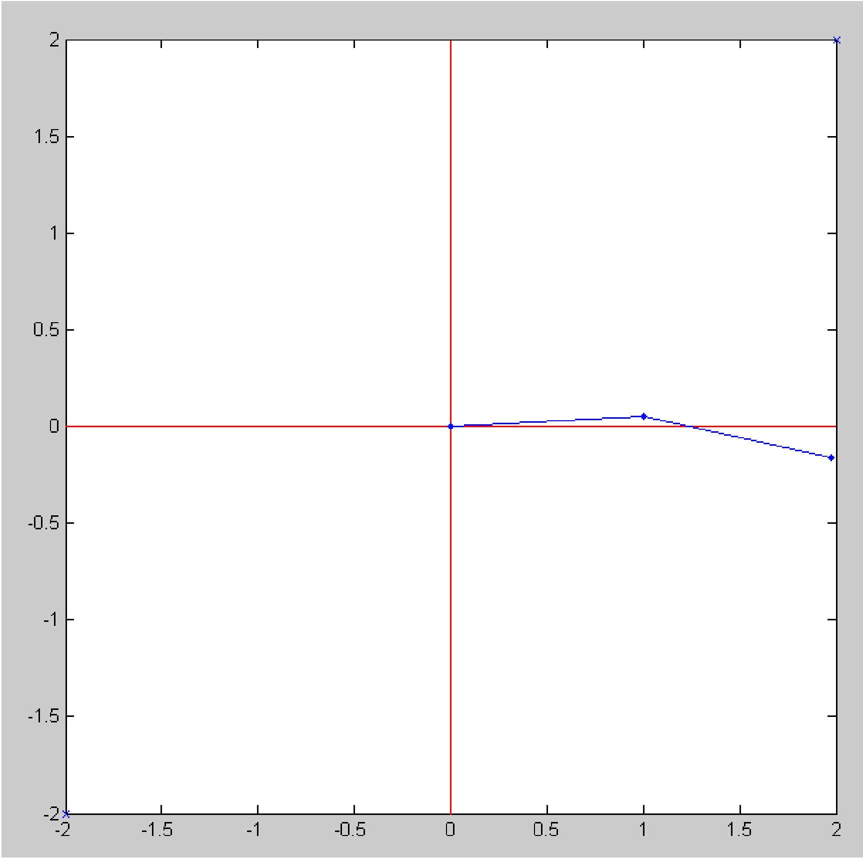
```
% 2-Link robot in free-fall (T=0)
% rev 6/5/20

Q = [1; 0];
dQ = [0; 0];
T = [0; 0];
t = 0;
dt = 0.01;
data = [];
while(t < 10)

   c1 = cos(Q(1));
   s1 = sin(Q(1));
   c2 = cos(Q(2));
   s2 = sin(Q(2));

% Freefall
   T = [0;0];

   ddQ = TwoLinkDynamics(Q, dQ, T);
   dQ = dQ + ddQ * dt;
   Q = Q + dQ*dt;
   t = t + dt;
```

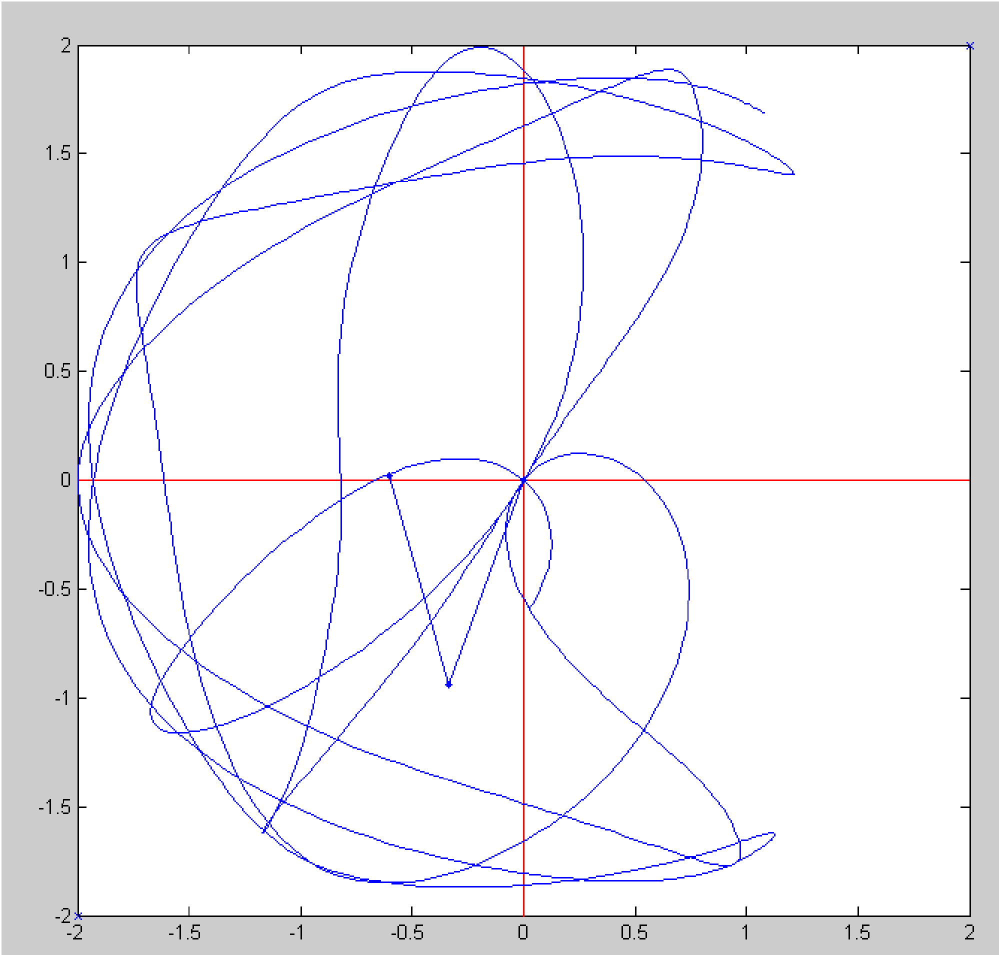


```
% Plot the Robot

    x0 = 0;
    y0 = 0;
    x1 = cos(Q(1));
    y1 = sin(Q(1));
    x2 = x1 + cos(Q(1) + Q(2));
    y2 = y1 + sin(Q(1) + Q(2));
    clf;
    plot([-2,2],[-2,2],'x');
    hold on;
    plot([-2,2],[0,0], 'r');
    plot([0,0],[-2,2], 'r');
    plot([y0, y1, y2], [x0, x1, x2], 'b.-');
    pause(0.01);
    data = [data ; x2, y2];
end
```

Note: This is a chaotic system

- Small changes in the initial conditions results in large changes in the trajectory



Butterfly Effect

https://en.wikipedia.org/wiki/Butterfly_effect

In chaos theory, the butterfly effect is the sensitive dependence on initial conditions in which a small change in one state of a deterministic nonlinear system can result in large differences in a later state.

The term butterfly effect is closely associated with the work of Edward Lorenz. It is derived from the metaphorical example of the details of a tornado (the exact time of formation, the exact path taken) being influenced by minor perturbations such as a distant butterfly flapping its wings several weeks earlier. Lorenz discovered the effect when he observed that runs of his weather model with initial condition data that were rounded in a seemingly inconsequential manner. He noted that the weather model would fail to reproduce the results of runs with the unrounded initial condition data. A very small change in initial conditions had created a significantly different outcome.[1]

Linearized Dynamics about (0,0)

$$\begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} = \begin{bmatrix} (3+2c_2) & (1+c_2) \\ (1+c_2) & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot\theta_1\dot\theta_2 + s_2\dot\theta_2^2 \\ -s_2\dot\theta_1^2 \end{bmatrix} + g\begin{bmatrix} 1s_1 + s_{12} \\ s_{12} \end{bmatrix} \right)$$

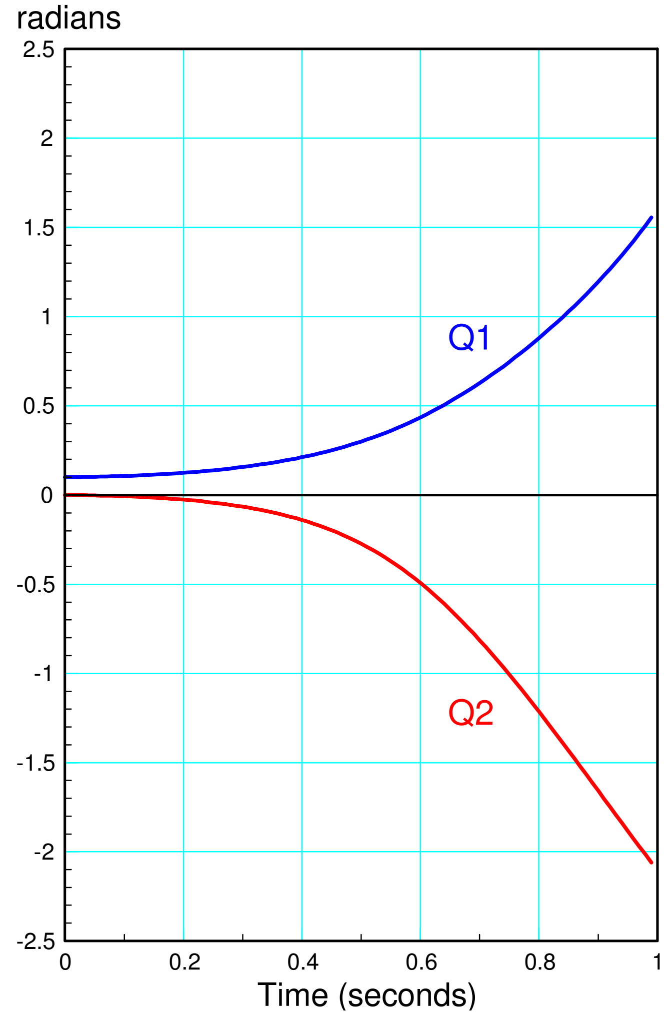
$$\begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} = +g\begin{bmatrix} -\theta_2 \\ \theta_1 + 3\theta_2 \end{bmatrix} \quad \Rightarrow \quad s\begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -g & 0 & 0 \\ g & 3g & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}$$

Free-Fall Simulation

- g = +9.8 m/s2

Eigenvalues

- -5.0652
- +5.0652
- -1.9348
- +1.9348



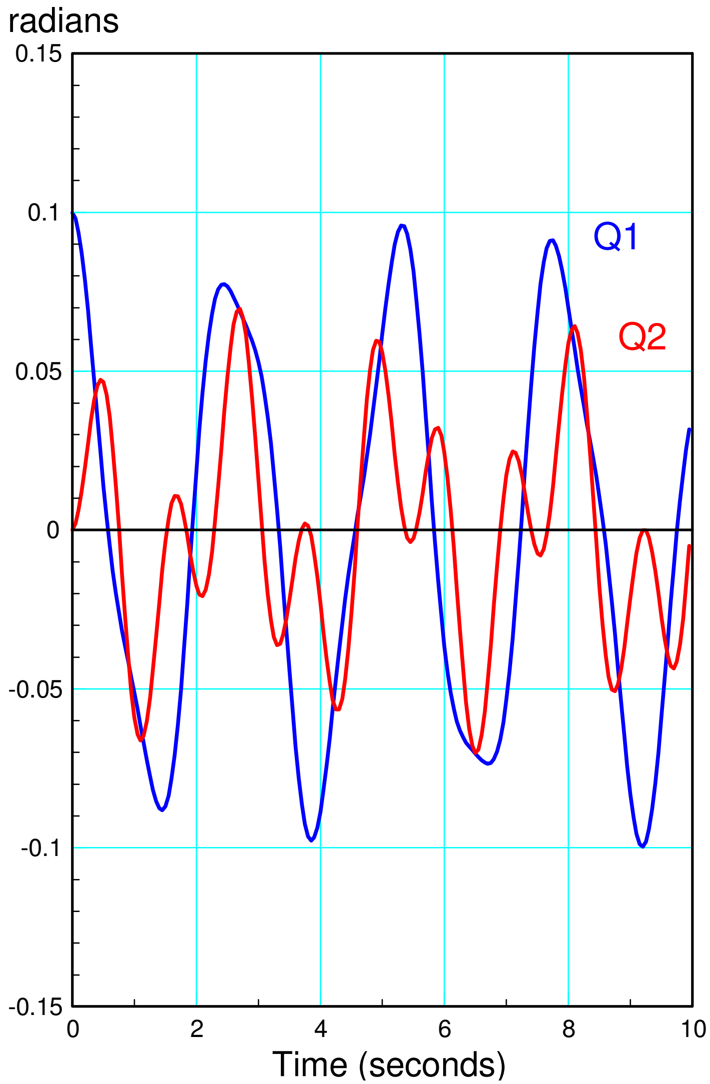
Free Fall Simulation

- g = -9.8 m/s2

Eigenvalues

```
eig(A)

  0 + 5.0652i
  0 - 5.0652i
  0 + 1.9348i
  0 + 1.9348i
```



Summary

LaGrangians are able to determine the dynamics of a 2-link arm

$$\begin{bmatrix} (3+2c_2) & (1+c_2) \\ (1+c_2) & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + g\begin{bmatrix} 2s_1 + s_{12} \\ s_{12} \end{bmatrix}$$

The same procedure is used for 3-link and 6-link robotic arms

- The derivation takes about 10 pages though.
- But, you only have to derive it once for a given robot.

Note: We spend considerable time in ECE 761 Robotics designing feedback controllers for this system.