Pole Placement (Bass Gura)

NDSU ECE 463/663

Lecture #13

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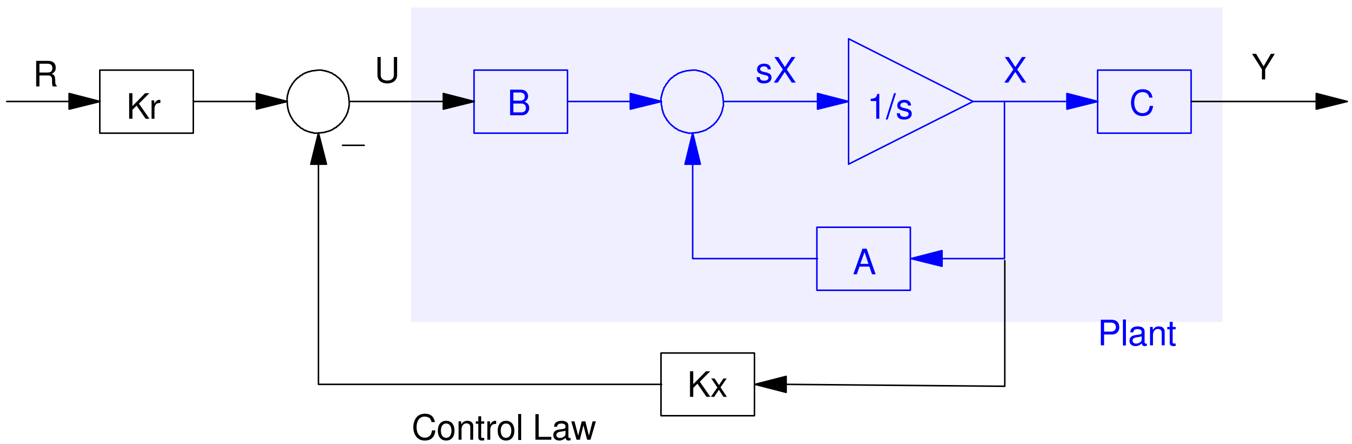
Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Pole Placement

If a system is controllable, Kx can place all N poles of the closed-loop system anywhere.

- How do you find Kx?
- How do you find Kr?



Definitions:

Open-Loop System: System dynamics with U = 0.

$$sX = AX$$

Closed-Loop System: System dynamics with U = -Kx X

$$sX = (A - BK_x)X$$

Characteristic Polynomial:

a) The polynomial with roots equal to the eigenvalues of A

```
poly(eig(A))
```

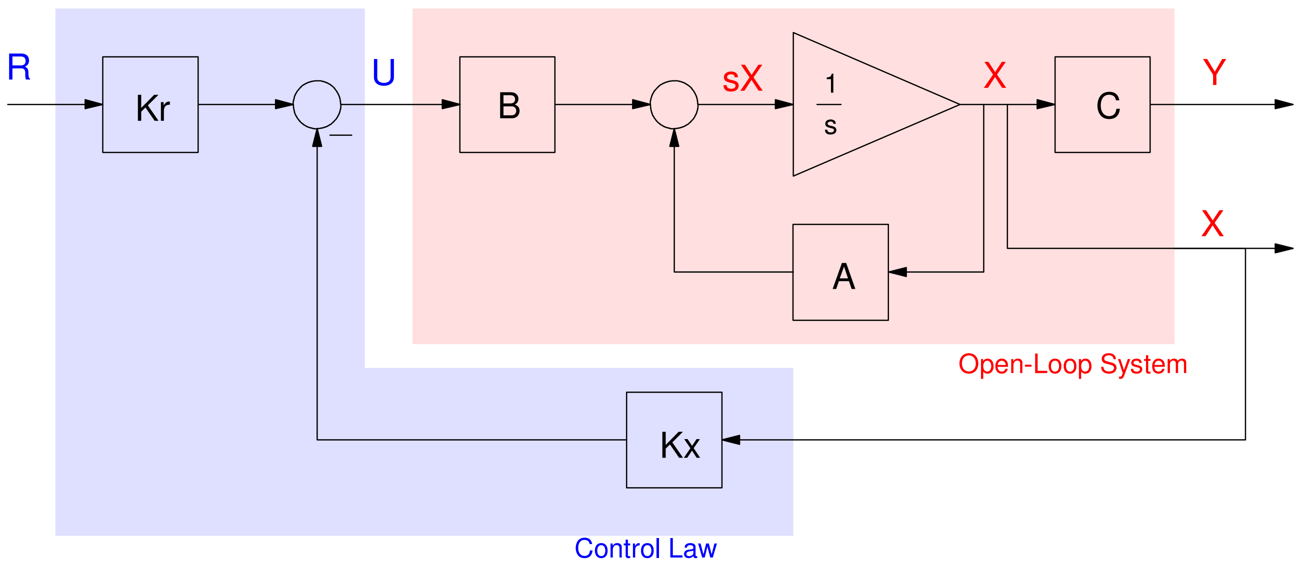
b) The denominator polynomial of the transfer function

Bass Gura Derivation:

Assume

- A system is controllable.
- Full-state feedback is used:

$$U = K_r R - K_x X$$



Problem: Find Kx and Kr to Place the Poles of the Closed-Loop System and Set the DC Gain from R to Y

Case 1: Controller Canonical Form:

Assume the system is in controller canonical form

- Characteristic polynomial (i.e. the denominator of the transfer function) is

$$P(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

Find the feedback gains so that the characteristic polynomial becomes

$$P_d(s) = s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0$$

In controller canonical form, the plant is:

$$sX = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} U$$

With full-state feedback:

$$U = -\begin{bmatrix} k_0 & k_1 & k_2 & k_3 \end{bmatrix} X$$

the dynamics become

$$sX = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 - k_0 & -a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 \end{bmatrix} X$$

By inspection, the closed-loop characteristic polynomial is:

$$s^4 + (a_3 + k_3)s^3 + (a_2 + k_2)s^2 + (a_1 + k_1)s + (a_0 + k_0) = 0$$

which is to be equal to the desired characteristic polynomial:

$$s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0 = 0$$

Match terms

- The feedback gains are the difference between the desired and open-loop characteristic polynomials.

$$k_3 = b_3 - a_3$$

$$k_2 = b_2 - a_2$$

$$k_1 = b_1 - a_1$$

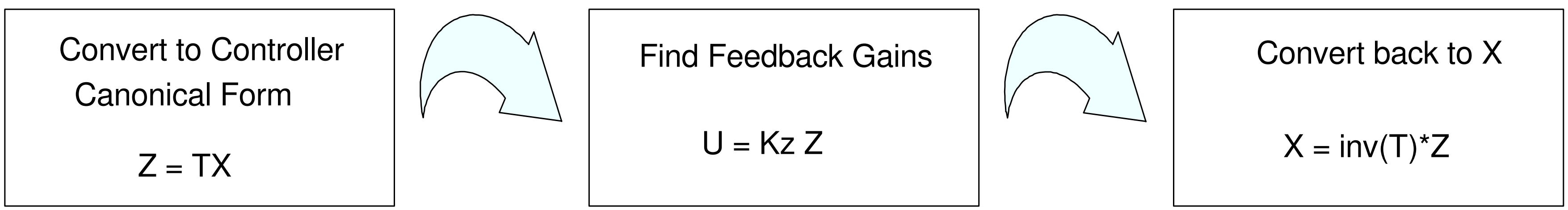
$$k_0 = b_0 - a_0$$

Case 2: The system not in controller canonical form

If this is the case,

- Find a similarity transform, T, which takes you to controller canonical form
- Find the feedback gains to place the closed-loop poles in controller form
- Convert these feedback gains to state-variable form with this similarity transform.

This method is called Bass-Gura or Pole Placement.



Bass Gura: Step 1

Find a similarity transform which takes you to controller canonical form.

$$T = T_1 T_2 T_3$$

T1 is the controllability matrix (assume a 4th-order system here):

$$T_1 = \left[\; B \; AB \; A^2B \; A^3B \;\right]$$

T2 is related to the system's characteristic polynomial

$$\det(sI - A) = s^4 + a_3 s^3 + a^2 s^2 + a_1 s + a_0$$

$$T_2 = \begin{bmatrix} 1 & a_3 & a_2 & a_1 \\ 0 & 1 & a_3 & a_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T3 is a flip matrix

$$T_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

This similarity transform results in the transformed system being

$$Z = TX$$

$$sZ = (T^{-1}AT)Z + (T^{-1}B)U$$

or

$$sZ = A_z Z + B_z U$$

where (Az, Bz) are in controller canonical form.

Note that since T includes the controllability matrix and T is inverted, the (A, B) must be controllable for this algorithm to work.

Bass Gura: Step 2

Determine the feedback gains in controller form (Kz)

- This is the difference between the desired and actual characeristic polynominals

Open-Loop Characteristic Polynomial:

$$P(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

Closed-Loop (desired) Characteristic Polynomial:

$$P_d(s) = s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0$$

Feedback Gains:

$$K_z = \left[\ (b_0 - a_0),\ (b_1 - a_1),\ (b_2 - a_2),\ (b_3 - a_3)\ \right]$$

Bass Gura: Step 3

Convert back to state-variable form (X) using the similarity transform:

$$K_x = K_z T^{-1}$$

Check your answer. The closed-loop system is then

$$sX = (A - BK_x)X$$

The eigenvalues of $(A - BK)$ should be where you wanted to place them.

Step 4: Set the DC Gain (Find Kr)

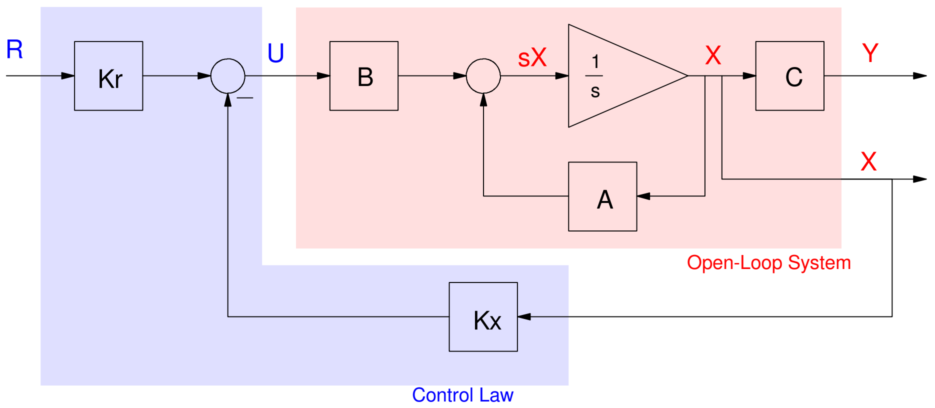
Assume

$$U = K_r R - K_x X$$

then the closed-loop dynamics are

$$sX = (A - BK_x)X + BK_r R$$

$$Y = CX$$



The steady-state (i.e. DC) gain is

$$sX = 0 = (A - BK_x)X + BK_r R$$

$$X = -(A - BK_x)^{-1} BK_r R$$

$$Y = -C(A - BK_x)^{-1} BK_r R$$

Pick Kr so that

$$-C(A - BK_x)^{-1} BK_r = 1$$

Example 1: Heat Equation.

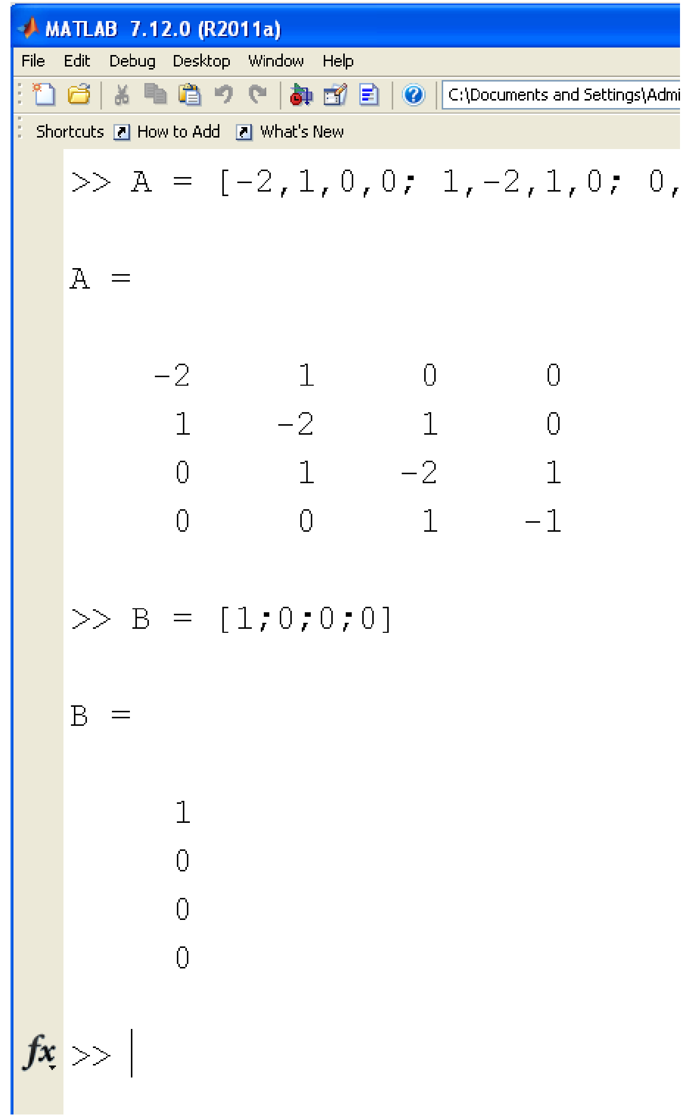
Assume a system has the following dynamics:

$$sX = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} U$$

Find the feedback gain, Kx, to place the poles of the closed-loop system

$$U = -K_x X$$

at { -1, -2, -3, -4}



Step 1: Find the similarity transform which takes you to controller canonical form.

T1 is the controllability matrix:

```
>> T1 = [B, A*B, A*A*B, A*A*A*B]

     1    -2     5   -14
     0     1    -4    14
     0     0     1    -6
     0     0     0     1

>> rank(T1)

ans =      4
```

The system is controllable

T2 is related to the system's characteristic polynomial

```
>> P = poly(eig(A))

    1.0000    7.0000   15.0000   10.0000    1.0000

>> T2 = [ P(1:4);  0, P(1:3);  0, 0, P(1:2);  0, 0, 0, P(1)]

    1.0000    7.0000   15.0000   10.0000
         0    1.0000    7.0000   15.0000
         0         0    1.0000    7.0000
         0         0         0    1.0000
```

T3 is a flip matrix

```
>> T3 = [0,0,0,1;0,0,1,0;0,1,0,0;1,0,0,0]

     0     0     0     1
     0     0     1     0
     0     1     0     0
     1     0     0     0
```

The transform to controller canonical form is:

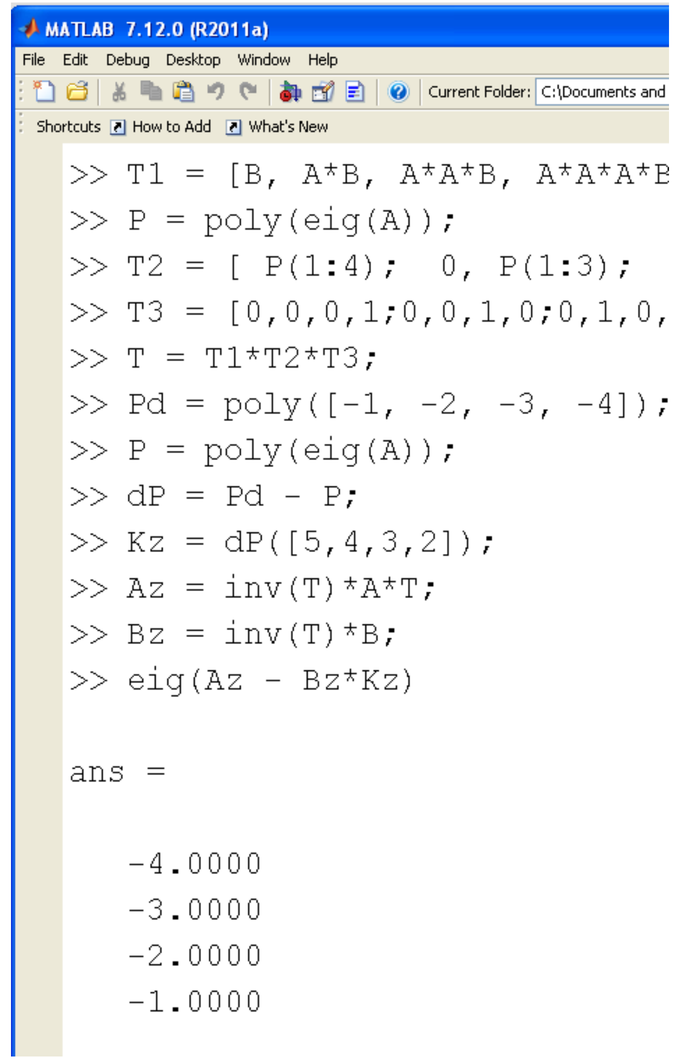
```
>> T = T1*T2*T3;

>> Az = inv(T)*A*T

         0    1.0000         0         0
         0   -0.0000    1.0000    0.0000
         0    0.0000    0.0000    1.0000
   -1.0000  -10.0000  -15.0000   -7.0000

>> Bz = inv(T)*B;

         0
         0
         0
         1
```

Check: (Az, Bz) are in controller canonical form.

Step 2: Find the full-state feedback gains in controller form.

- The difference between the desired and open-loop characteristic polynomials:

```
>> Pd = poly([-1, -2, -3, -4])

     1    10    35    50    24

>> P = poly(eig(A))

     1     7    15    10     1

>> dP = Pd - P

     0     3    20    40    23

>> Kz = dP([5,4,3,2])

    23    40    20     3
```



Step 3: Convert Kz to the gain times the state variables (X)

```
>> Kx = Kz*inv(T)

    3.0000    5.0000    7.0000    8.0000

>> eig(A - B*Kx)

   -4.0000
   -3.0000
   -2.0000
   -1.0000
```

The control law which places the closed-loop poles at { -1, -2, -3, -4 } is

$$U = -K_x X$$

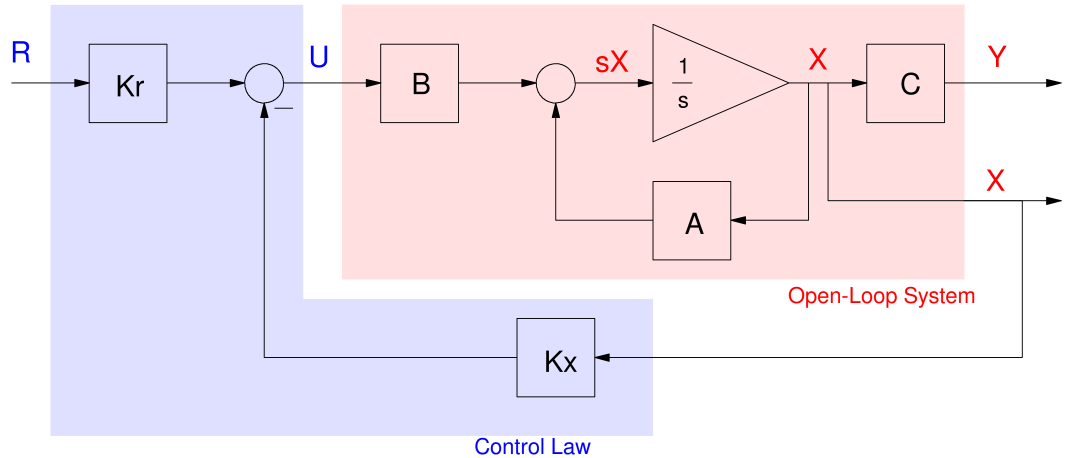
where

$$K_x = \left[\ 3\ 5\ 7\ 8\ \right]$$

Finding Kr:

$$-C(A-BK_x)^{-1}BK_r = 1$$

```
>> DC = -C*inv(A-B*Kx)*B

DC =     0.0417

>> Kr = 1/DC

Kr =    24.0000
```



So the control law is

$$U = K_r R - K_x X$$

$$K_r = 24$$

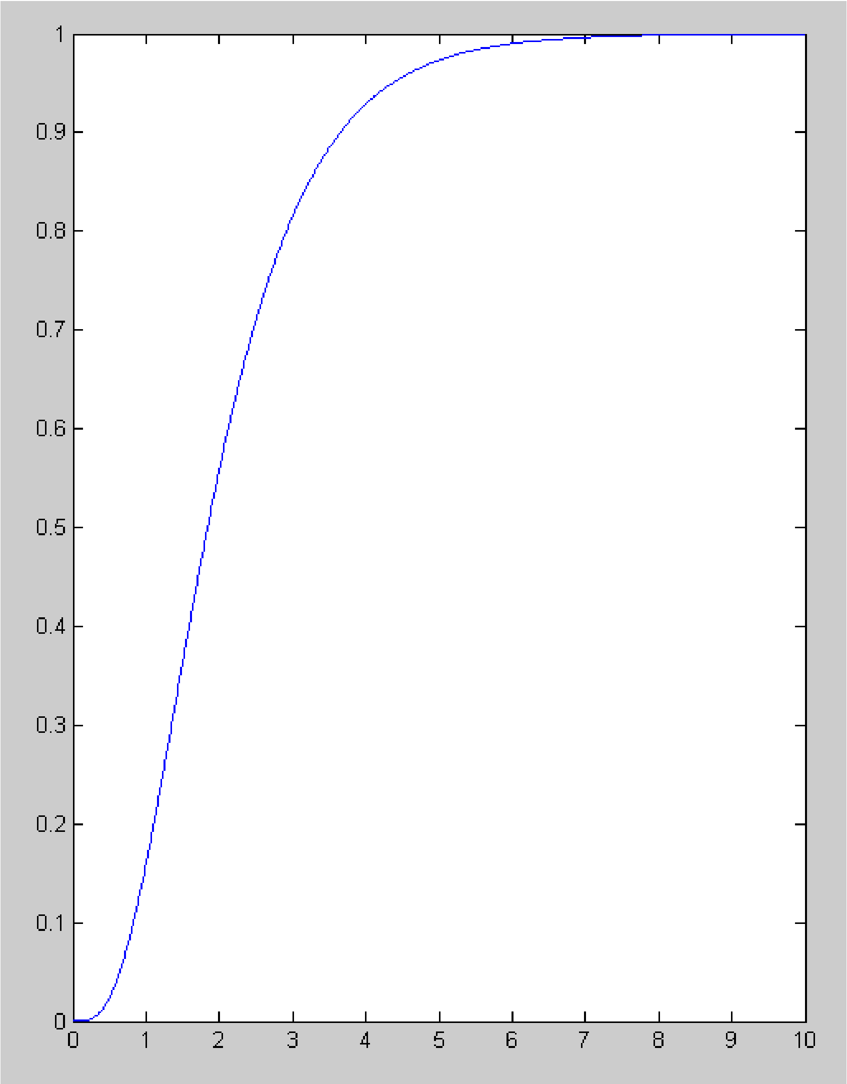
$$K_x = \left[\ 3\ 5\ 7\ 8\ \right]$$

Plotting the Step Response

```
>> t = [0:0.01:10]';
>> G = ss(A-B*Kx, B*Kr, C, 0);
>> y = step(G,t);
>> plot(t,y);
```

Note:

- The DC gain is 1.000
- The 2% settling time is 4ish seconds
  - Dominant pole is at -1
- No overshoot for step input
  - Dominant pole is real



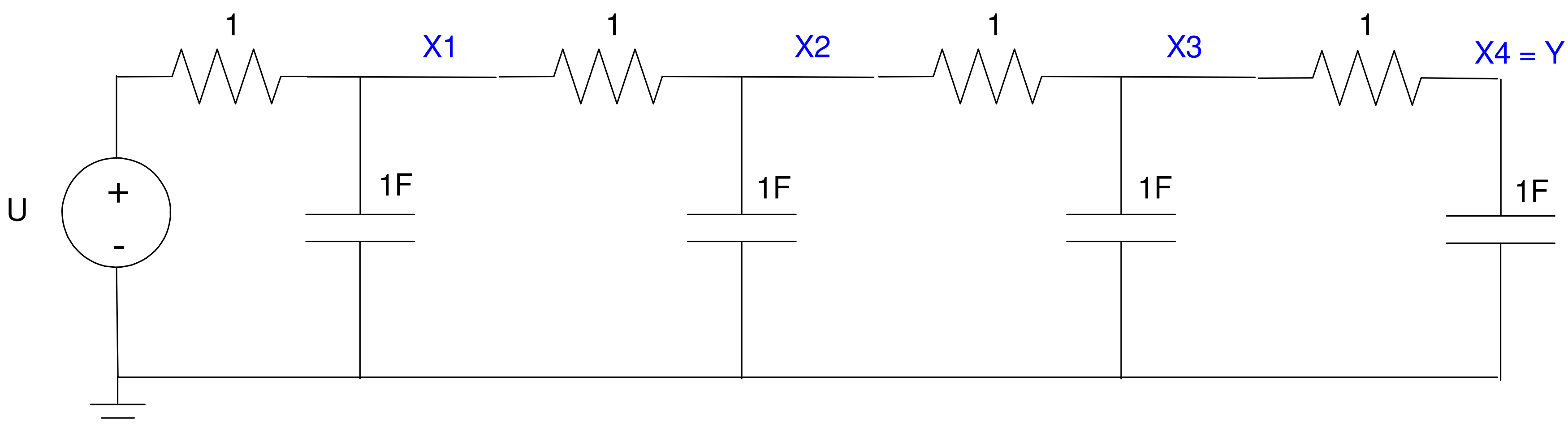
Example 2: Complex Poles

You can place the poles *anywhere* with pole placement

You can even make an RC circuit oscillate

Example: Place the closed-loop poles at

$$\{ -1 + j3,\ -1 - j3,\ -5 + j2,\ -5 - j2 \}$$



Bass Gura

Step 0: Input the system (done)

Step 1: Find the similarity transform (done)

Step 2: Find the feedback gains, Kz

```
>> Pd = poly([-1 + j*3,  -1 - j*3,  -5 + j*2,  -5-j*2])

     1    12    59   158   290

>> P = poly(eig(A))

    1.0000    7.0000   15.0000   10.0000    1.0000

>> dP = Pd - P

     0     5    44   148   289

>> Kz = dP([5,4,3,2])

   289   148    44     5
```

Step 3: Convert Kz to Kx:

```
>> Kx = Kz*inv(T)

    5.0000   19.0000   61.0000  204.0000
```

Check Kx:

```
>> eig(A - B*Kx)

  -5.0000 + 2.0000i
  -5.0000 - 2.0000i
  -1.0000 + 3.0000i
  -1.0000 - 3.0000i
```

Find Kr

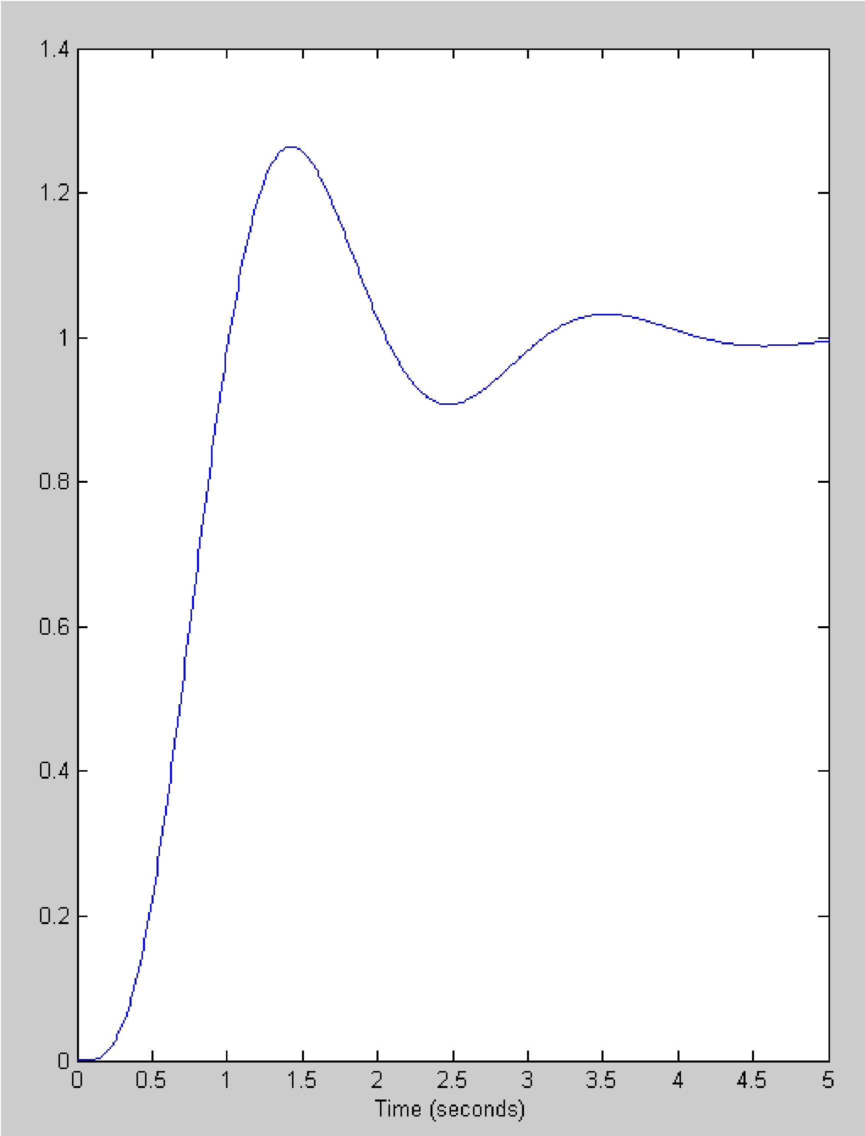
```
>> DC = -C*inv(A-B*Kx)*B

DC =     0.0034

>> Kr = 1/DC

Kr =   290.0000
```

So the resulting control law is:

$$U = K_r R - K_x X$$

$$K_r = 290$$

$$K_x = \begin{bmatrix} 5 & 19 & 61 & 204 \end{bmatrix}$$

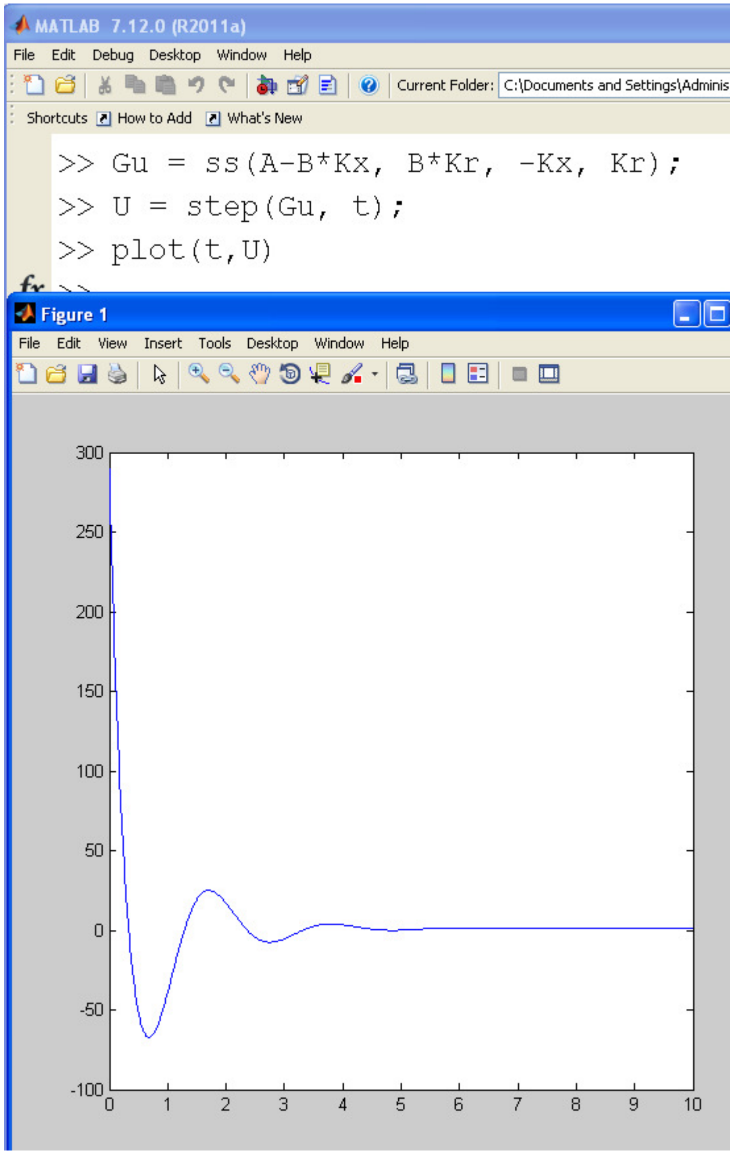


How is this possible?

In Thrust we Trust

Raw power.

- Large feedback gains tell you you're trying to force the system to behave contrary to its nature
- In theory, it can be done
- In practice, it takes a lot of power



How is this used?

https://asian-defence.blogspot.com/2011/05/russia-experts-says-russia-could-sell.html
http://iptmajor.weebly.com/uploads/3/8/3/2/38326191/2319116_orig.jpg

With full-state feedback, any response is possible

- You can make a MIG behave like an F-14
- You can make a research-grade robot behave like an assembly-line robot

Bass Gura allows you to find full-state feedback gains to place the closed-loop poles *anywhere*.