Full-Order Observers

NDSU ECE 463/663

Lecture #19

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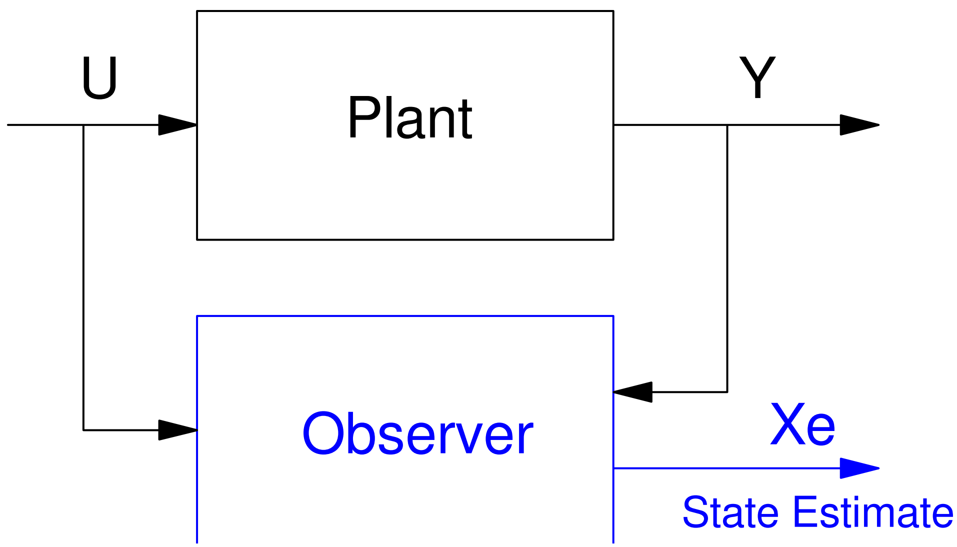
Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Problem:

- Pole Placement requires knowledge of *all* of the states
- What if some of the states cannot be measured?

Can you estimate the states based upon

- The system input,
- The system output, and
- The dynamics of the system?



Full-Order Observer Derivation

Given a system with states X:

$$sX = AX + BU$$

$$Y = CX$$

Define a model of the system

- State Estimate: Xe

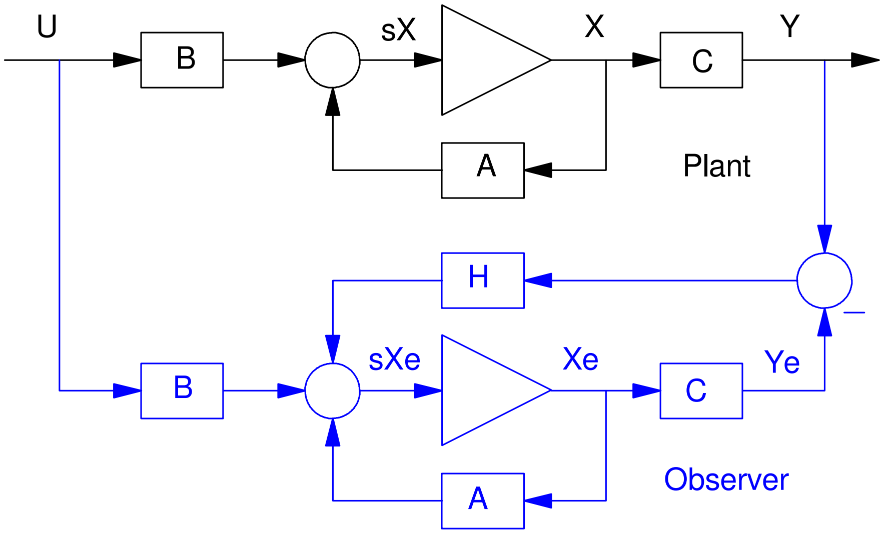
$$sX_e = AX_e + BU + H(Y - Y_e)$$

$$Y_e = CX_e$$

Define the error between the two

$$E = X - X_e$$

If $E \to 0$ then $X_e \to X$



Take the difference between the dynamics

$$sX - sX_e = (AX + BU) - (AX_e + BU) - H(Y - Y_e)$$

Do some algebra to find the dynamics of the error:

$$s(X - X_e) = A(X - X_e) + BU - BU - H(CX - CX_e)$$

$$sE = AE - HCE$$

$$sE = (A - HC)E$$

Note:

- If you can make (A - HC) stable, the error will be driven to zero.
- E is non controllable: It is not affected by U. (good)

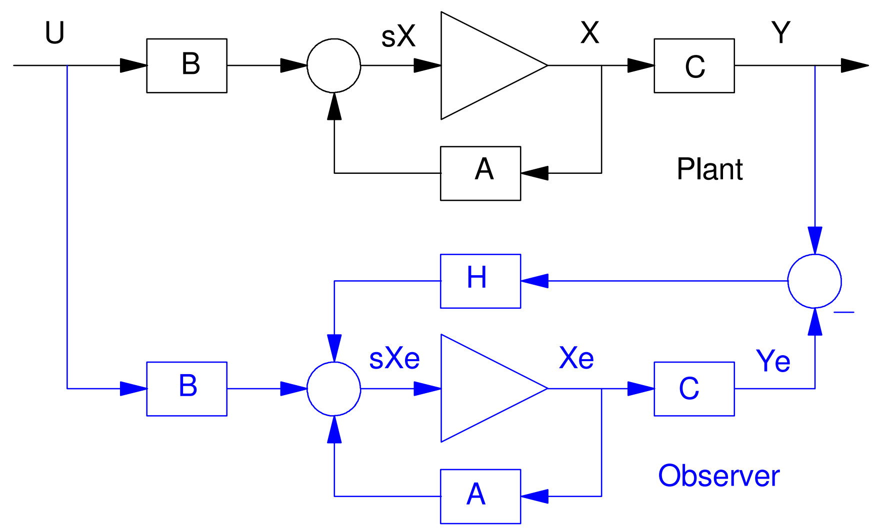
How to find H?

Pole placement is one way

- Pick H to place the poles of (A - HC)

Pole placement was designed for (A - B Kx)

- Change the problem to fit the solution



Pick H to place the poles of $(A - HC)^T$

$$(A - HC)^T = (A^T - C^T H^T)$$

Use Bass Gura with

- A^T for A
- C^T for B
- H^T for Kx

Limitations:

Bass Gura only works if the system is controllable:

$$\rho[B, AB, A^2B, \ldots, A^{N-1}B] = N$$

Substituting (A^T, C^T) for (A, B) results in

$$\rho\left[C^T, A^TC^T, \ldots (A^T)^{N-1}C^T\right] = N$$

or

$$\rho\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix} = N$$

This is termed the *observability* matrix.

- The system must be observable if you're using Bass Gura to find H.

Once you're done, the system becomes

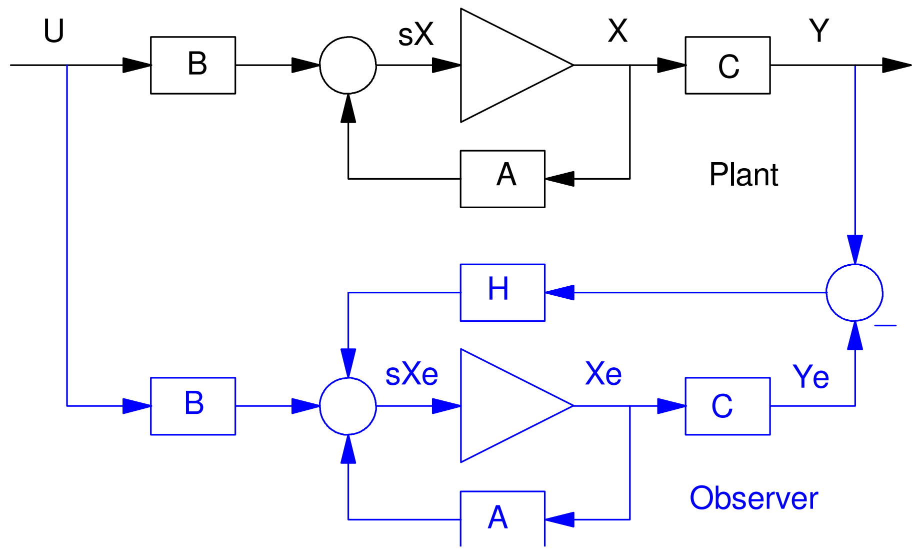
$$sX = AX + BU$$

$$sX_e = AX_e + BU + H(Y - Y_e)$$

Combining, the augmented system (Plant & Observer) becomes:

$$s\begin{bmatrix} X \\ X_e \end{bmatrix} = \begin{bmatrix} A & 0 \\ HC & A - HC \end{bmatrix}\begin{bmatrix} X \\ X_e \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix}U$$

$$Y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} X \\ X_e \end{bmatrix}$$



Example 1:

Assume you can only measure the 4th state for a heat equation. Design a full-order observer to estimate all four states

$$sX = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} X$$

Solution: First, check that the system is observable (i.e. that it can be done)

```
-->rank([C; C*A; C*A*A; C*A*A*A])

    4.
```

Place the observer poles at {-1, -2, -3, -4}

```
-->H = ppl(A', C', [-1, -2, -3, -4])'

    1.
    2.
    2.
    3.

-->eig(A-H*C)

  - 1.
  - 2.
  - 4.
  - 3.
```

The augmented system (plant plus observer) is then

```
-->A8 = [A, zeros(4,4); H*C, A-H*C]

  - 2.    1.    0.    0. :   0.    0.    0.    0.
    1.  - 2.    1.    0. :   0.    0.    0.    0.
    0.    1.  - 2.    1. :   0.    0.    0.    0.
    0.    0.    1.  - 1. :   0.    0.    0.    0.
   - - - - - - - - - - - - - - - - - - - - - - - - -
    0.    0.    0.    1. : - 2.    1.    0.  - 1.
    0.    0.    0.    2. :   1.  - 2.    1.  - 2.
    0.    0.    0.    2. :   0.    1.  - 2.  - 1.
    0.    0.    0.    3. :   0.    0.    1.  - 4.

-->B8 = [B; B]
 B8  =

    1.
    0.
    0.
    0.
  - - -
    1.
    0.
    0.
    0.
```

The poles of the augmented system are the poles of the plant (A) and the observer poles:

```
-->eig(A8)

  - 0.1206148
  - 4.
  - 3.5320889
  - 3.
  - 2.3472964
  - 2.
  - 1.
  - 1.
```

Note that the augmented system is uncontrollable:

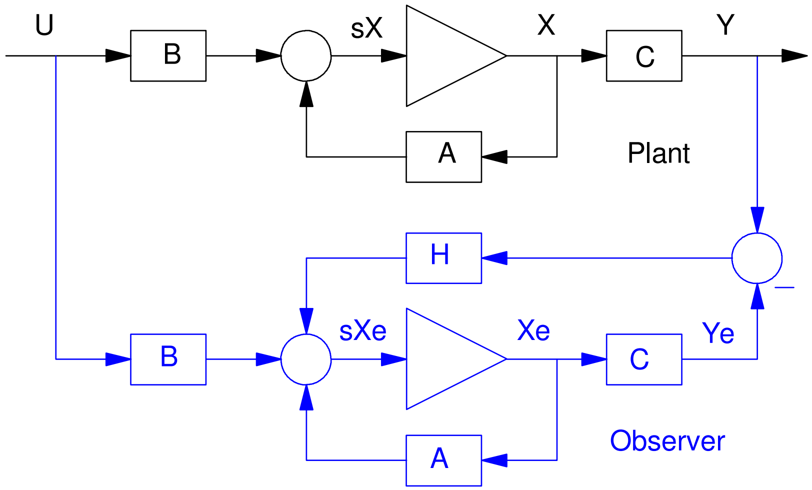
```
rank([B8,A8*B8,(A8^2)*B8,(A8^3)*B8,(A8^4)*B8,(A8^5)*B8,(A8^6)*B8,(A8^7)
*B8])

    4.
```

That's actually expected: the error in the observer states goes to zero regardless of what the input, U, does. The error in the observer estimates is uncontrollable. (that's good).

Simulate the following 8x8 system with initial conditions:

$$s\begin{bmatrix} X \\ X_e \end{bmatrix} = \begin{bmatrix} A & 0 \\ HC & A-HC \end{bmatrix}\begin{bmatrix} X \\ X_e \end{bmatrix}$$

```
>> X0 = [0;0;0;0;  0.2;0.4;0.6;0.8]

         0
         0      plant at t= 0
         0
         0
 - - - - -
    0.2000
    0.4000      observer at t=0
    0.6000
    0.8000
```



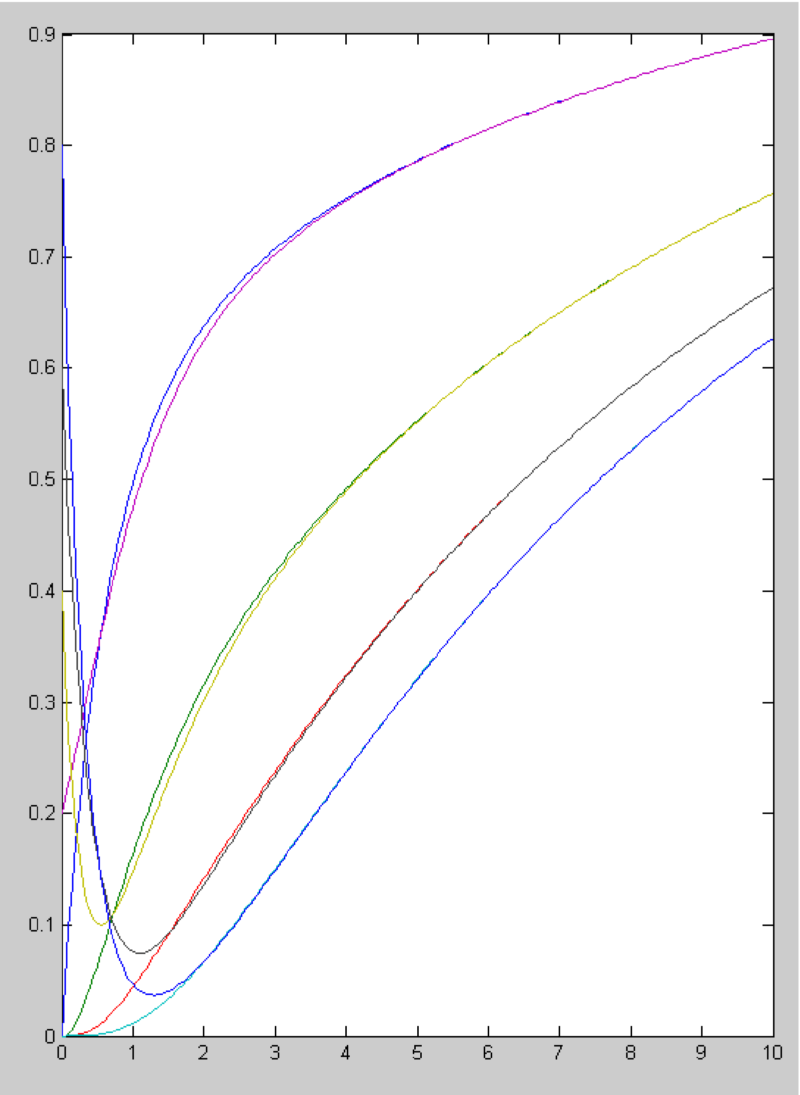
Output all eight states:

```
C8 = eye(8,8);
D8 = zeros(8,1);
```

Simulate using the function step2:

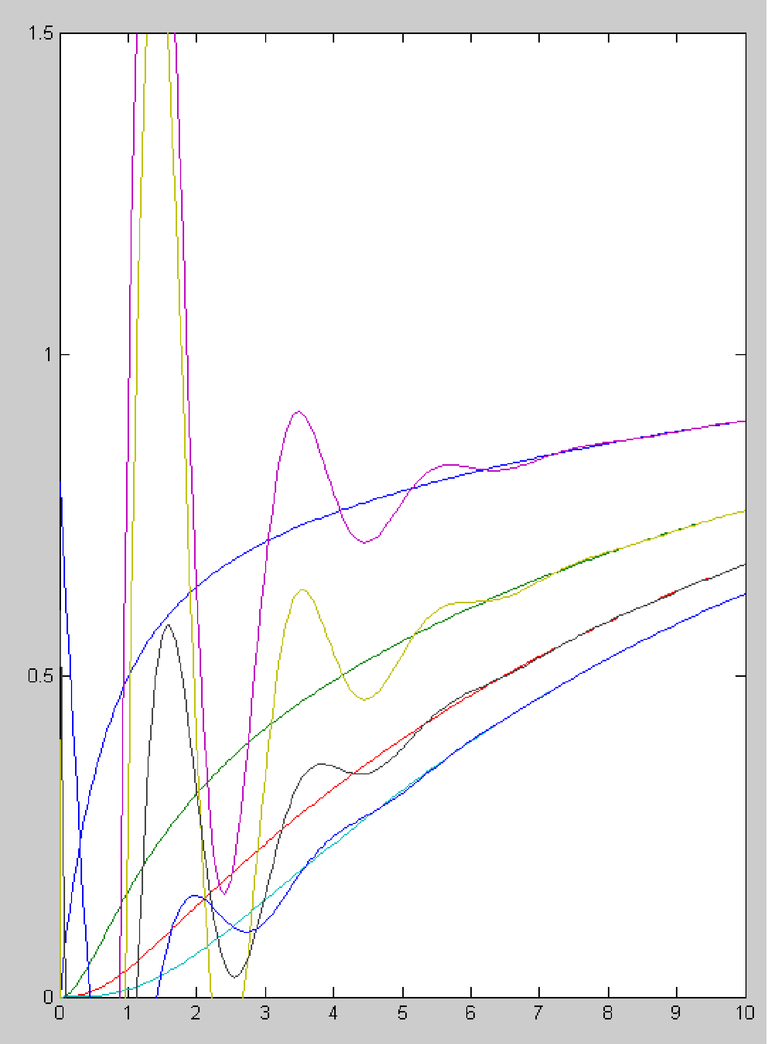
```
R = 0*t + 1;
y = step3(A8, B8, C8, D8, t, X0, R);
plot(t,y);
```

Note that the state estimates (Xe) converge to the actual states (X)



You can even make the observer poles complex if you like:

```
H=ppl(A',C',[-1+j*3,-1-j*3,-3, -4])'

     28.
     28.
     9.
     2.

A8 = [A, zeros(4,4); H*C, A-H*C]
B8 = [B; B]
C8 = eye(8,8);
D8 = zeros(8,1);

X0 = [0;0;0;0;  0.2;0.4;0.6;0.8]
y = step3(A8, B8, C8, D8, t, X0, R);
plot(t,y);
```



Again, the observer states (Xe) converge to the plant states (X)

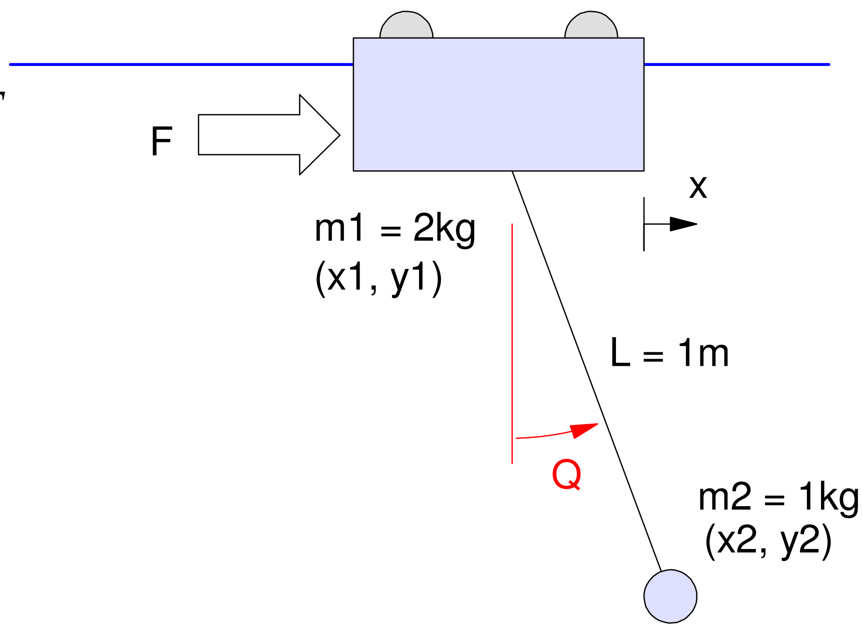
Example 2: Gantry System

Can you stabilize a gantry system with only one measurement?

- If the system is observable, then yes.

Design a full-order observer for a gantry system:

$$s\begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 4.9 & 0 & 0 \\ 0 & -14.7 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix}F$$



Option 1: Use measurement of angle (θ)

$$Y = \theta = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix}$$

Check that the system is observable from angle:

```
A = [0,0,1,0;0,0,0,1;0,4.9,0,0;0,-14.7,0,0]
C = [0,1,0,0]
rank([C; C*A; C*A*A; C*A*A*A])

    2.
```

You cannot estimate all four states just by measuring only the angle.

Option 2: Use measurements of position (x)

$$Y = x = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix}$$

Is the system observable?

```
A = [0,0,1,0;0,0,0,1;0,4.9,0,0;0,-14.7,0,0]
C = [1,0,0,0]

    1.    0.    0.    0.

rank([C; C*A; C*A*A; C*A*A*A])

    4.
```

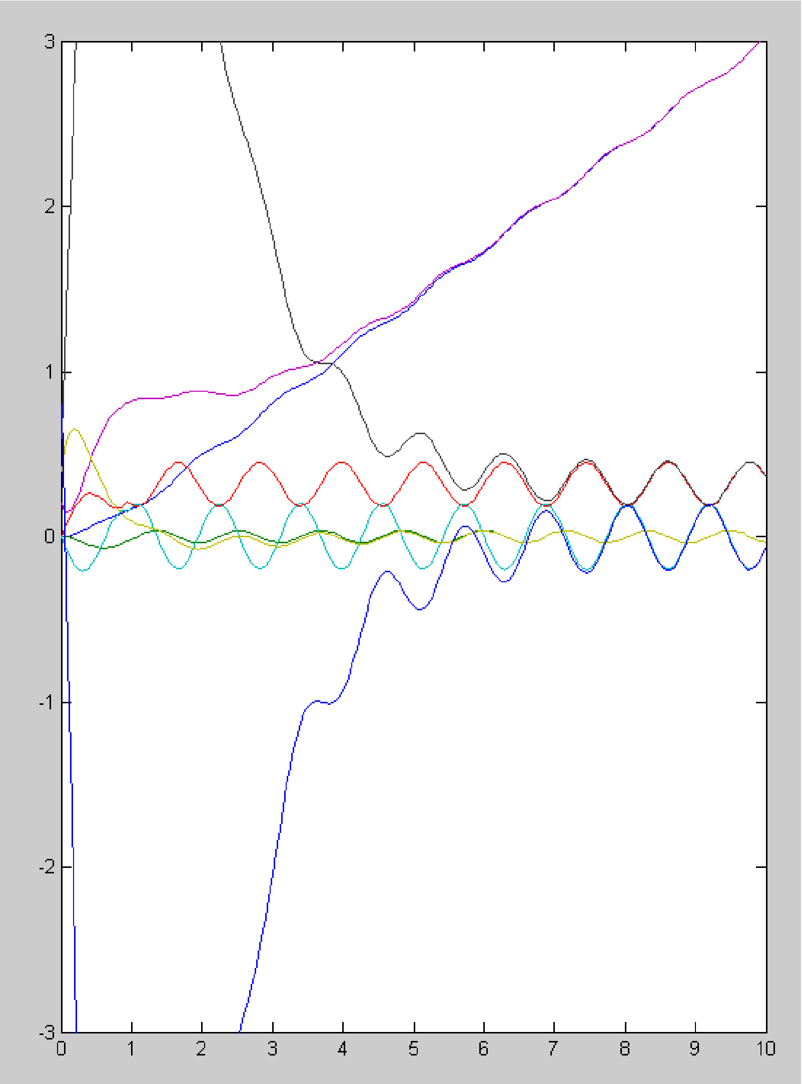
Yes. There is enough information in the position measurement to

- Estimate the other three states, and
- Control the system.

Observer Design:

```
H = ppl(A', C', [-1, -2, -3, -4])'

   10.0000
  -19.7959
   20.3000
  -56.0020

A8 = [A, zeros(4,4) ; H*C, A-H*C ]

          plant                       model
      0        0        1        0 :      0        0       0       0
      0        0        0        1 :      0        0       0       0
      0     4.90        0        0 :      0        0       0       0
      0   -14.70        0        0 :      0        0       0       0
--------------------------------------------------------------
   10.00       0        0        0 :  -10.00       0       1       0
  -19.79       0        0        0 :   19.79       0       0       1
   20.30       0        0        0 :  -20.30    4.90       0       0
  -56.00       0        0        0 :   56.00  -14.70       0       0
```

```
X0 = [0;0;0;0;  0.2;0.4;0.6;0.8]

          0
          0    X(0)
          0
          0
- - - - - - -
     0.2000
     0.4000    Xe(0)
     0.6000
     0.8000


R = 1*(t < 1);
y = step3(A8, B8, C8, D8, t, X0, R);
plot(t,y);
```



Note: The observer states converge to the plant states

Summary:

If the system is observable from the output, you can estimate *all* of the system states using only

- The system input (U),
- The system output (Y), and
- The dynamics of the system (A, B, C, D)

The results is termed a *State Estimator* or *Full Order Observer*