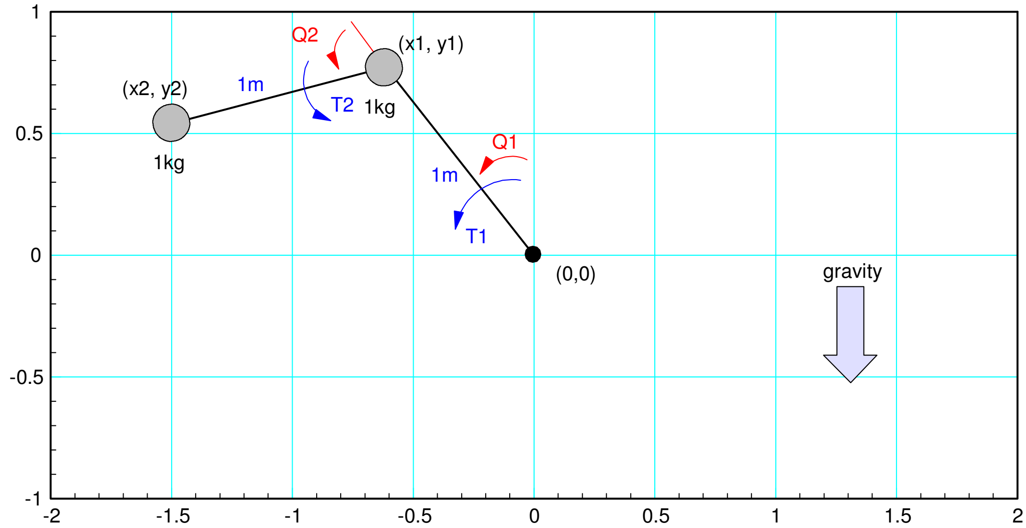
Dynamics of a 2-Link Arm



Problem: Determine the dynamics of a 2-link arm with

- Each arm having a length of 1m, and
- Each arm has a lump-mass of 1kg at the end of each arm.

Step 1: Determine the (x,y) position and velocity of each mass

mass 1:

$$x_1 = -\sin(\theta_1) \qquad \dot{x}_1 = -\cos(\theta_1)\dot{\theta}_1$$

$$y_1 = \cos(\theta_1) \qquad \dot{y}_1 = -\sin(\theta_1)\dot{\theta}_1$$

mass 2:

$$x_2 = x_1 - \sin(\theta_1 + \theta_2) \qquad \dot{x}_2 = -\cos(\theta_1)\dot{\theta}_1 - \cos(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$y_2 = y_1 + \cos(\theta_1 + \theta_2) \qquad \dot{y}_2 = -\sin(\theta_1)\dot{\theta}_1 - \sin(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

Using shorthand notation

$$x_1 = -s_1 \qquad \dot{x}_1 = -c_1\dot{\theta}_1$$

$$y_1 = c_1 \qquad \dot{y}_1 = -s_1\dot{\theta}_1$$

$$x_2 = -s_1 - s_{12} \qquad \dot{x}_2 = -c_1\dot{\theta}_1 - c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$y_2 = c_1 + c_{12} \qquad \dot{y}_2 = -s_1\dot{\theta}_1 - s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

Step 2: Form the kinetic and potential energy:

Mass 1:

$$KE = \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right)$$

$$KE = \tfrac{1}{2}\left(\left(-c_1\dot{\theta}_1\right)^2 + \left(-s_1\dot{\theta}_1\right)\right)$$

Note that $\sin^2 + \cos^2 = 1$

$$KE = \tfrac{1}{2}\dot{\theta}_1^2$$

$$PE = mgy_1$$

$$PE = gc_1$$

Mass 2:

$$KE = \tfrac{1}{2}m\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$KE = \tfrac{1}{2}\left(\left(-c_1\dot{\theta}_1 - c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)^2 + \left(-s_1\dot{\theta}_1 - s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)^2\right)$$

$$KE = \tfrac{1}{2}\left(c_1^2\dot{\theta}_1^2 + c_{12}^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + s_1^2\dot{\theta}_1^2 + s_{12}^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2\right)$$

$$c_1c_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + s_1s_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$KE = \tfrac{1}{2}\left(\dot{\theta}_1^2 + \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2\right) + c_1c_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + s_1s_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

From trigenomietry

$$\sin(a)\sin(b) + \cos(a)\cos(b) = \cos(a-b)$$

resulting in

$$KE = \tfrac{1}{2}\left(\dot{\theta}_1^2 + \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2\right) + c_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$KE = (1 - c_2)\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + (1 + c_2)\dot{\theta}_1\dot{\theta}_2$$

$$PE = m_2gy_2$$

$$PE = g(c_1 + c_{12})$$

Step 3: Form the LaGrangian:

$$L = KE - PE$$

$$L = \left((1+c_2)\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + (1+c_2)\dot{\theta}_1\dot{\theta}_2\right) - (gc_1 + g(c_1 + c_{12}))$$

$$\boxed{L = (1+c_2)\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + (1+c_2)\dot{\theta}_1\dot{\theta}_2 - 2gc_1 - gc_{12}}$$

Step 4: Take the partials

$$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right)$$

$$T_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right)$$

Starting with θ_1:

$$\frac{\partial L}{\partial \dot{\theta}_1} = 2(1+c_2)\dot{\theta}_1 + (1+c_2)\dot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 2(1+c_2)\ddot{\theta}_1 + (1+c_2)\ddot{\theta}_2 - 2s_2\dot{\theta}_1\dot{\theta}_2 - s_2\dot{\theta}_2^2$$

$$\frac{\partial L}{\partial \theta_1} = 2gs_1 + gs_{12}$$

Putting it together:

$$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right)$$

$$\boxed{T_1 = 2(1+c_2)\ddot{\theta}_1 + (1+c_2)\ddot{\theta}_2 - 2s_2\dot{\theta}_1\dot{\theta}_2 - s_2\dot{\theta}_2^2 - (2gs_1 + gs_{12})}$$

Correct Equations from John Craig

$$T_1 = 2(1+c_2)\ddot{\theta}_1 + (1+c_2)\ddot{\theta}_2 - 2s_2\dot{\theta}_1\dot{\theta}_2 - s_2\dot{\theta}_2^2 - (2gc_1 + gc_{12})$$

Moving on to θ_2

$$L = (1+c_2)\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + (1+c_2)\dot{\theta}_1\dot{\theta}_2 - 2gc_1 - gc_{12}$$

Take the partial derivative with respect to dq2:

$$\frac{\partial L}{\partial \dot{\theta}_2} = \dot{\theta}_2 + (1+c_2)\dot{\theta}_1$$

Taking the full derivative:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \ddot{\theta}_2 + (1+c_2)\ddot{\theta}_1 - s_2\dot{\theta}_1\dot{\theta}_2$$

Now, on the partial with respect to q2:

$$\frac{\partial L}{\partial \theta_2} = -s_2\dot{\theta}_1^2 - s_2\dot{\theta}_1\dot{\theta}_2 + gs_{12}$$

Putting it together:

$$T_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right)$$

$$T_2 = \left(\ddot{\theta}_2 + (1+c_2)\ddot{\theta}_1 - s_2\dot{\theta}_1\dot{\theta}_2\right) - \left(-s_2\dot{\theta}_1^2 - s_2\dot{\theta}_1\dot{\theta}_2 + gs_{12}\right)$$

$$\boxed{T_2 = \ddot{\theta}_2 + (1+c_2)\ddot{\theta}_1 + s_2\dot{\theta}_1^2 - gs_{12}}$$

Correct Equations from John Craig:

$$T_2 = \ddot{\theta}_2 + (1+c_2)\ddot{\theta}_1 + s_2\dot{\theta}_1^2 + gc_{12}$$

Net result:

$$T_1 = 2(1+c_2)\ddot{\theta}_1 + (1+c_2)\ddot{\theta}_2 - 2s_2\dot{\theta}_1\dot{\theta}_2 - s_2\dot{\theta}_2^2 + 2gc_1 + gc_{12}$$

$$T_2 = \ddot{\theta}_2 + (1+c_2)\ddot{\theta}_1 + s_2\dot{\theta}_1^2 + gc_{12}$$

To solve, rewrite this in matrix form:

$$\begin{bmatrix} 2(1+c_2) & (1+c_2) \\ (1+c_2) & 1 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} - g\begin{bmatrix} 2c_1 + c_{12} \\ c_{12} \end{bmatrix}$$

Mass Matrix * Acceleration = Torque - Coriolis Forces + Gravity

Given the joint angles, velocities, gravity, and input torques, you can compute the joint accelerations as

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 2(1-c_2) & (1-c_2) \\ (1-c_2) & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} - \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} 2gs_1 + gs_{12} \\ gs_{12} \end{bmatrix} \right)$$

Note that the mass matrix is zero when θ_2 is 180 degrees. This suggests that the acceleration goes to infinity at this position (and should be avoided)

MatLab Code:

```
function [ ddQ ] = TwoLinkDynamics( Q, dQ, T )

q1 = Q(1);
q2 = Q(2);

dq1 = dQ(1);
dq2 = dQ(2);

c1 = cos(q1);
s1 = sin(q1);
s2 = sin(q2);
c2 = cos(q2);
s12 = sin(q1+q2);
c12 = cos(q1+q2);

g = 9.8;

M = [ 2*(1+c2) + 0.002,     1+c2 ;
       1+c2            ,   1+0.002   ];


C = [ 2*s2*dq1*dq2 + s2*dq2*dq2 ;
           -s2*dq1*dq1          ];

G = [2*g*s1 + g*s12 ;
          g*s12    ];

ddQ = inv(M) * ( T + C + G );


end
```

Note: The mass matrix becomes singular at 0 and 180 degrees. To avoid this singularity, 0.01 is added to the diagonal (the arm isn't exactly a point mass)

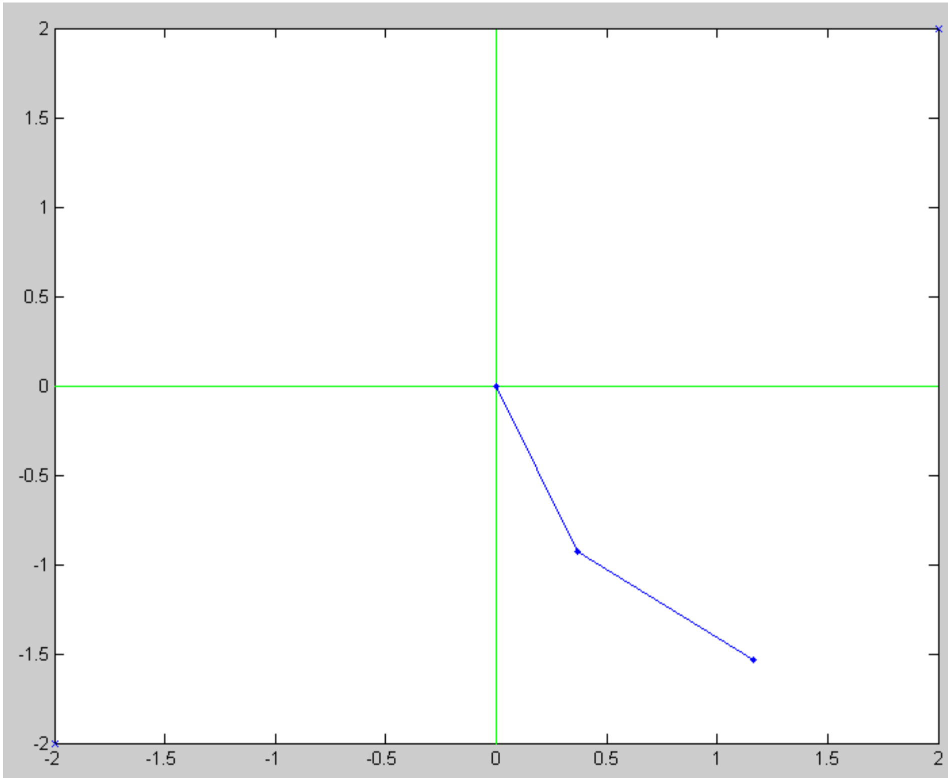


Image of 2-link arm in free-fall

Main Calling Routine:

```
Q = [2; 0.2];
dQ = [0; 0];

T = [0; 0];

t = 0;
dt = 0.01;

while(t < 100)

   ddQ = TwoLinkDynamics(Q, dQ, T);

   dQ = dQ + ddQ * dt;
   Q = Q + dQ*dt;
   t = t + dt;


   x0 = 0;
   y0 = 0;

   x1 = -sin(Q(1));
   y1 =  cos(Q(1));

   x2 = x1 - sin(Q(1) + Q(2));
   y2 = y1 + cos(Q(1) + Q(2));

   clf;
   plot([-2,2],[-2,2],'x');
   hold on;
   plot([-2,2],[0,0], 'g');
   plot([0,0],[-2,2], 'g');

   plot([x0, x1, x2], [y0, y1, y2], 'b.-');
   pause(0.01);

   end
```