Rotation Matricies

Lecture #2

ECE 761: Robotics

Class taught at North Dakota State University
Department of Electrical and Computer Engineering

Please visit www.BisonAcademy.com for corresponding lecture notes, homework sets, and solutions.

Rotation Matrices

As the robot moves, often times the angle of each frame changes. Likewise, in order to go from one link to the next, a rotation and translation is required.

This lecture looks at rotation matrices.

In the {x, y, z} plane, you can rotate about any or all axis. For simplicity, we will look at three separate rotations:

- Rotating about the X axis
- Rotating about the Y axis, and
- Rotating about the Z axis.

In addition, we will be referring to a point (P) relative to a specific reference frame. P1, for example, refers to the {x, y, z} coordinate of point P relative to reference frame 1.

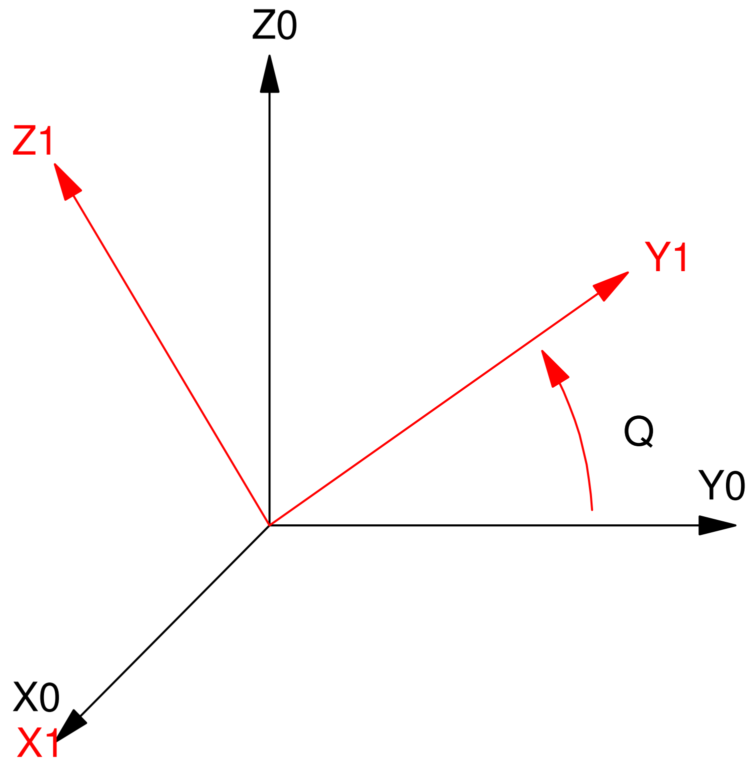
Rotation About the X Axis

Find the position of point P0, relative to frame 1

- Rotate Frame #0 by θ about it's x-axis to get to frame 1.

$$P_1 = T_{10} \cdot P_0$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$



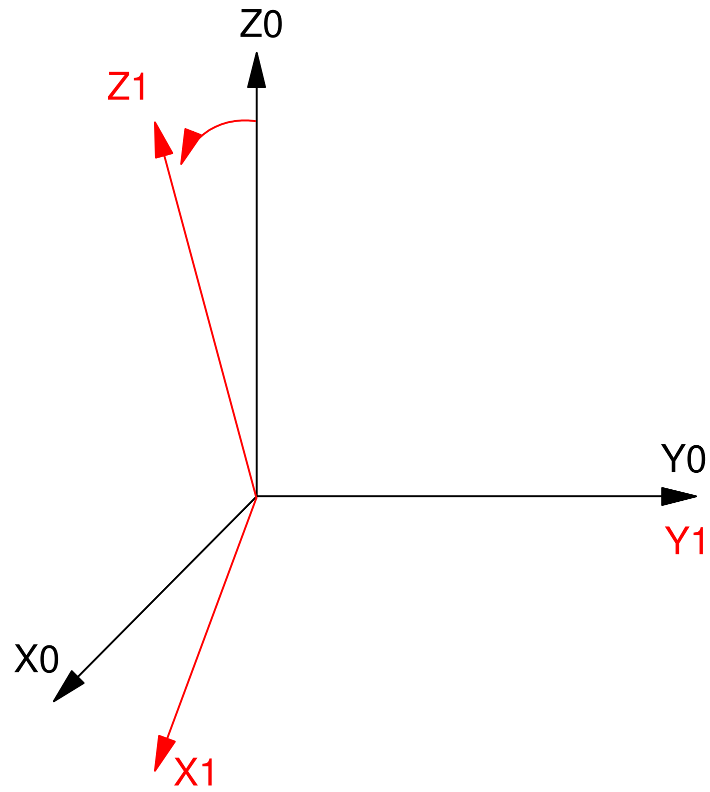
Rotate About the Y-Axis:

Find the position of point P0, relative to frame 1

- Rotate Frame #0 by θ about it's y-axis to get to frame 1.

$$P_1 = T_{10} \cdot P_0$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$



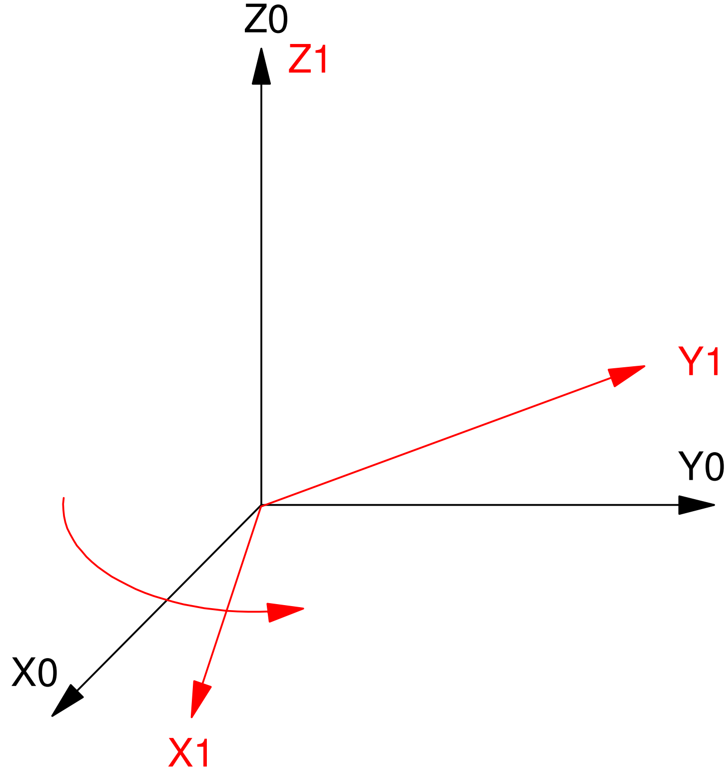
Rotate about the z-axis

Find the position of point P0, relative to frame 1

- Rotate Frame #0 by θ about it's y-axis to get to frame 1.

$$P_1 = T_{10} \cdot P_0$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$



Example: A point relative to the zero reference frame is (1,2,3).

$$P_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$$

Where is it at if you

- Rotate 90 degrees about the X axis
- Rotate 90 degrees about the X axis
- Rotate 90 degrees about the X axis

Solution: For convenience, define c and s to be the sine and cosine of 90 degrees:

```
c = cos(90*pi/180);
s = sin(90*pi/180);
```

From the above equations

```
Rx = [1,0,0,0;0,c,s,0;0,-s,c,0;0,0,0,1];
Ry = [c,0,-s,0;0,1,0,0;s,0,c,0;0,0,0,1];
Rz = [c,s,0,0;-s,c,0,0;0,0,1,0;0,0,0,1];

P0 = [1;2;3;1]

Rx * P0
Ry * P0
Rz * P0
```

No Rotation	X Axis	Y Axis	Z Axis
1 2 3	1 3 -2	-3 2 1	2 -1 3

Example 2: Where is the point if you

- Rotate 20 degrees about the Y axis, then
- Rotate 30 degrees about the Z axis?

Define three transformations. The point relative to reference frame 0 is

$$P_1 = T_{10} \cdot P_0 = \begin{bmatrix} \cos 20^0 & 0 & \sin 20^0 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin 20^0 & 0 & \cos 20^0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$$

$$P_2 = T_{21} \cdot P_1 = \begin{bmatrix} \cos 30^0 & -\sin 30^0 & 0 & 0 \\ \sin 30^0 & \cos 30^0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot P_1$$

in MATLAB

```
c = cos(20*pi/180);
s = sin(20*pi/180);

Rx = [1,0,0,0;0,c,s,0;0,-s,c,0;0,0,0,1];
Ry = [c,0,-s,0;0,1,0,0;s,0,c,0;0,0,0,1];
Rz = [c,s,0,0;-s,c,0,0;0,0,1,0;0,0,0,1];

T10 = Ry;

c = cos(30*pi/180);
s = sin(30*pi/180);

Rx = [1,0,0,0;0,c,s,0;0,-s,c,0;0,0,0,1];
Ry = [c,0,-s,0;0,1,0,0;s,0,c,0;0,0,0,1];
Rz = [c,s,0,0;-s,c,0,0;0,0,1,0;0,0,0,1];

T21 = Rz;

P0 = [1;2;3;1]
P1 = T10 * P0
P2 = T21 * P1
```

The point relative to the zero reference frame is:

```
P0 =

     1
     2
     3
     1
```

The point relative to the first reference frame (rotated 20 degrees about the y axis)

```
P1 =

   -0.0864
    2.0000
    3.1611
    1.0000
```

The point relative to the second reference frame (finally rotated 30 degrees about the z axis)

```
P2 =

    0.9252
    1.7752
    3.1611
    1.0000
```

In the rotated coordinate system, the point is located at $P_2 = \begin{pmatrix} 0.9252 \\ 1.7752 \\ 3.1611 \end{pmatrix}$.

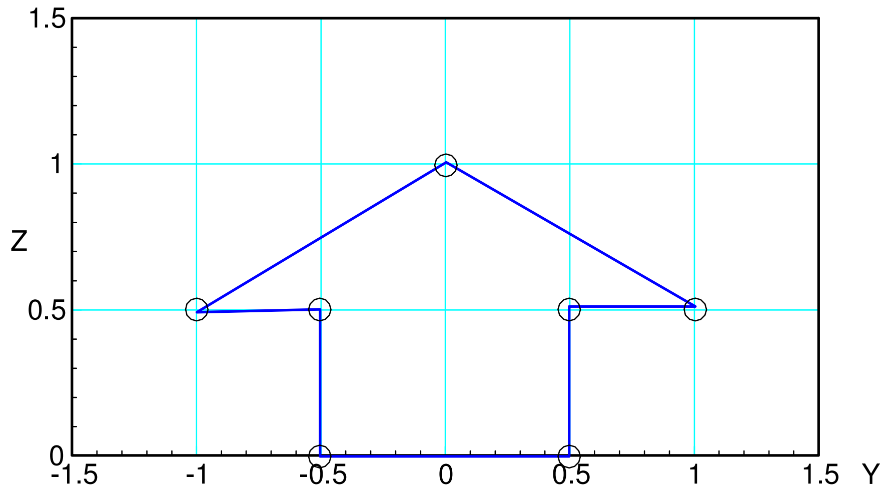
Displaying a 3D Object

To illustrate the use of rotation matrices, let's draw an arrow and then

- Rotate the camera about the X, Y, and Z axis, then
- Rotate the arrow about the X, Y, and Z axis

Define the arrow by eight points:

$$Arrow = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0.5 & 1 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0 \end{bmatrix}$$

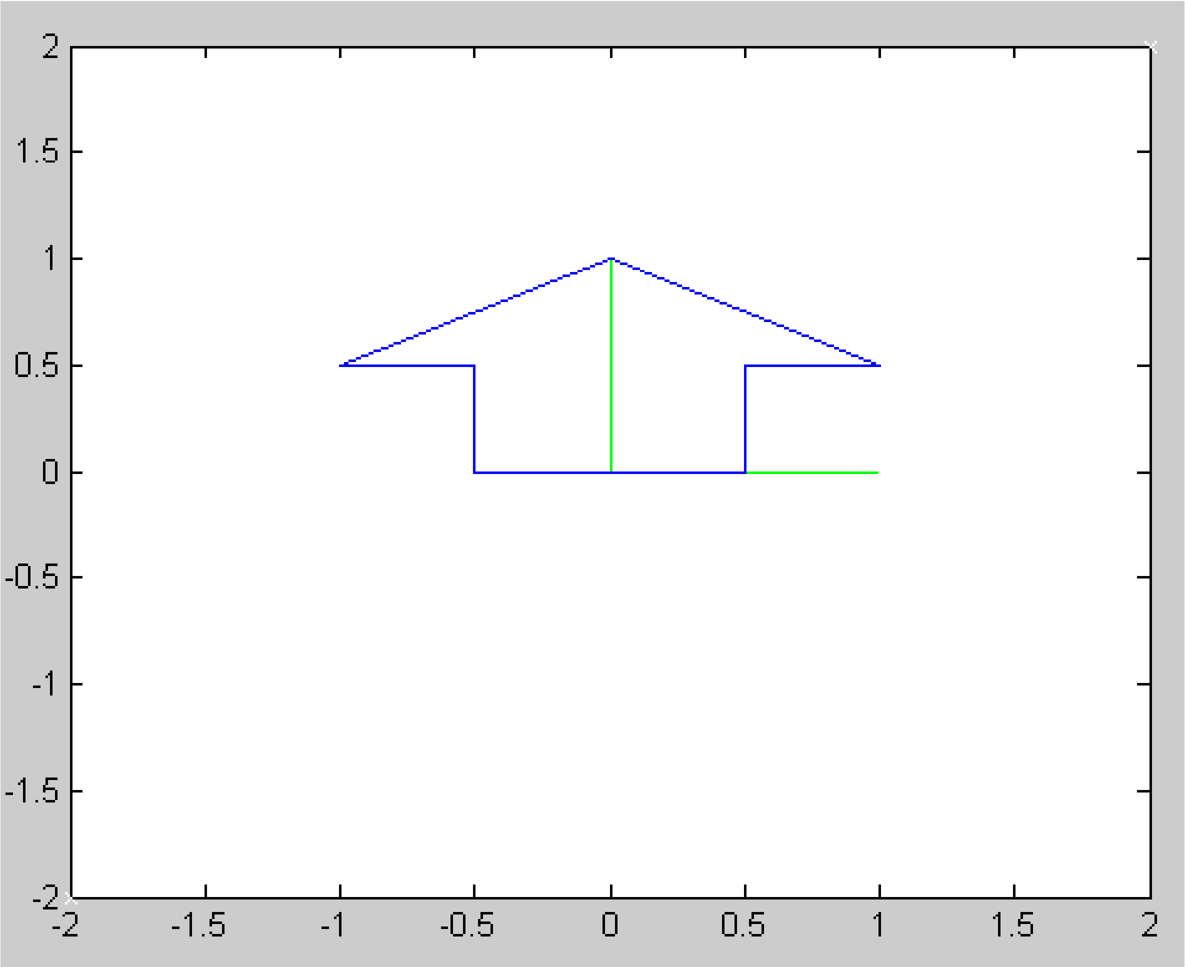


Project the arrow on the YZ plane. An m-file to display a set of points (passed in DATA) along with a transformation matrix is as follows:

```
function  Display3D(DATA, T)
```

The previous arrow looks like the following:

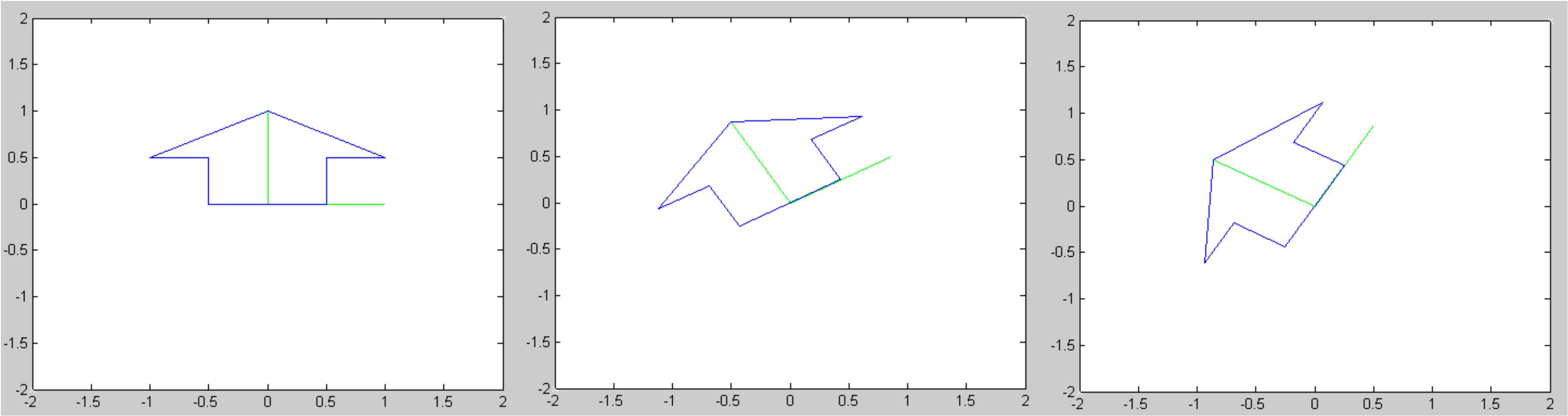
```
X = [0,0,0,0,0,0,0,0]';
Y = [-0.5,0.5,0.5,1,0,-1,-0.5,-0.5]';
Z = [0,0,0.5,0.5,1,0.5,0.5,0]';

ARROW = [X,Y,Z,0*X+1]'

        0         0         0         0         0         0         0         0
  -0.5000    0.5000    0.5000    1.0000         0   -1.0000   -0.5000   -0.5000
        0         0    0.5000    0.5000    1.0000    0.5000    0.5000         0
   1.0000    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000

T = eye(4,4);
Display3D(ARROW, T);
```



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

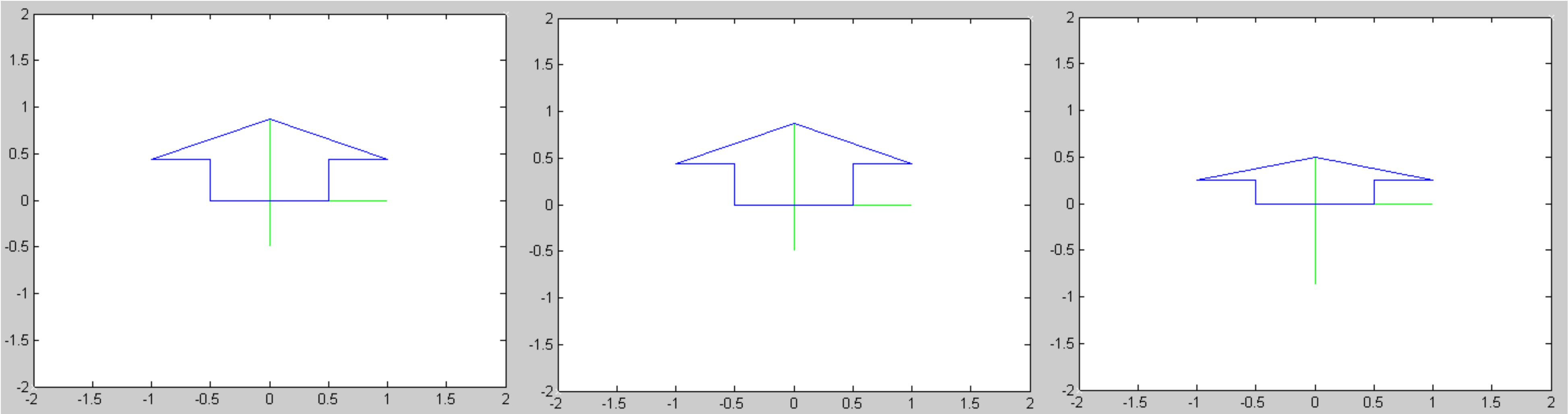
Rotate the camera about the X axis

```
c = cos(5*pi/180);
s = sin(5*pi/180);
Rx = [1,0,0,0;0,c,s,0;0,-s,c,0;0,0,0,1];

T = eye(4,4);
for i=1:1000
   T = Rx*T;
   Display3D(ARROW, T);
   pause(0.01);
   end
```



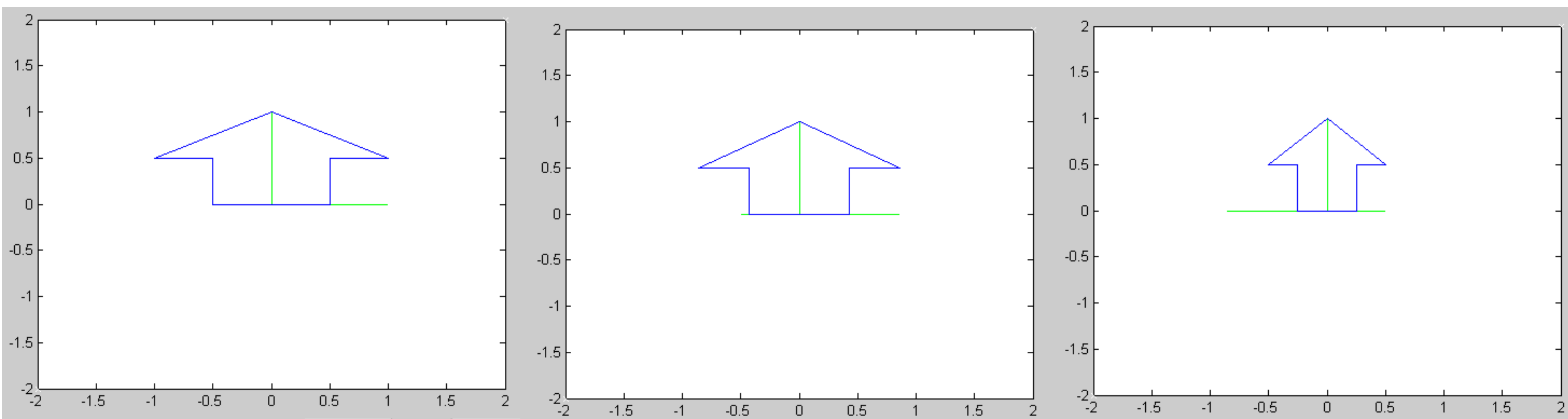
Rotate the Camera About the Y Axis

```
Ry = [c,0,-s,0;0,1,0,0;s,0,c,0;0,0,0,1];
    0.9962         0    0.0872         0
         0    1.0000         0         0
   -0.0872         0    0.9962         0
         0         0         0    1.0000
T = eye(4,4);
for i=1:1000
   T = Ry*T;
   Display3D(ARROW, T);
   pause(0.01);
   end
```



Rotate the Camera About the Z Axis

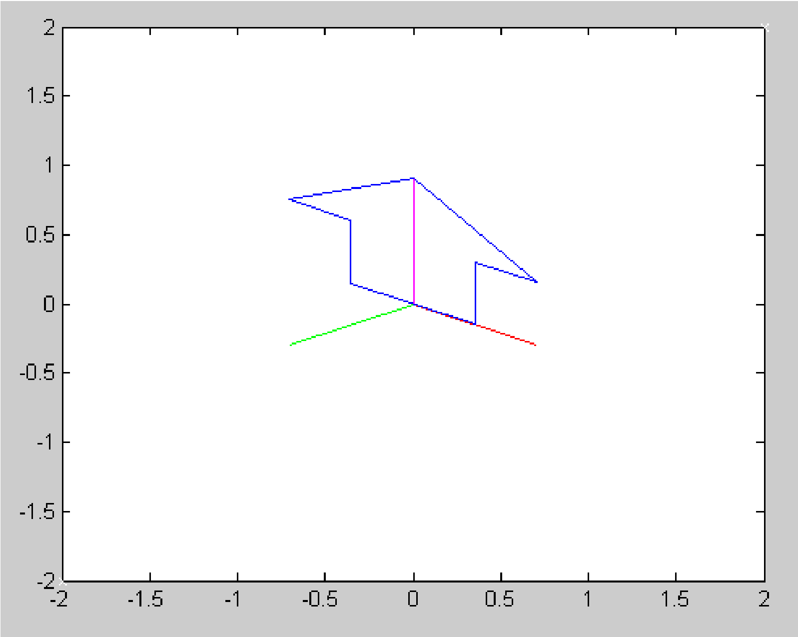
```
Rz = [c,-s,0,0;s,c,0,0;0,0,1,0;0,0,0,1]
   0.9962   -0.0872         0         0
   0.0872    0.9962         0         0
        0         0    1.0000         0
        0         0         0    1.0000
T = eye(4,4);
for i=1:1000
   T = Tz*T;
   Display3D(ARROW, T);
   pause(0.01);
   end
```



3D Perspective:

Move the camera 45 degrees about the Z axis then 25 degrees about the Y axis

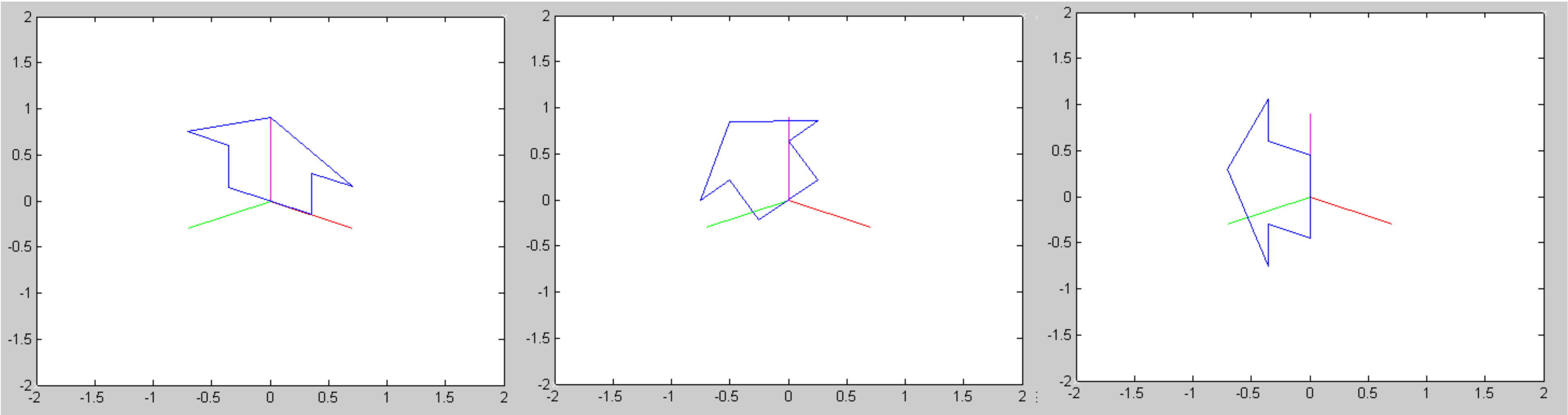
```
c = cos(-25*pi/180);
s = sin(-25*pi/180);
Ry = [c,0,-s,0;0,1,0,0;s,0,c,0;0,0,0,1];
c = cos(45*pi/180);
s = sin(45*pi/180);
Rz = [c,-s,0,0;s,c,0,0;0,0,1,0;0,0,0,1]
Tdisp = Ry*Rz
Display3D(ARROW,Tdisp);
```



Rotate the Arrow About the X Axis

You can also rotate the object while keeping the axis fixed

```
c = cos(5*pi/180);
s = sin(5*pi/180);
Rx = [1,0,0,0;0,c,s,0;0,-s,c,0;0,0,0,1];
T01 = inv(Rx);

T = eye(4,4);
for i=1:1000
   T = T01*T;
   Display3D(T*ARROW, Tdisp);
   pause(0.01);
   end
```



Note: The arrow positions are relative to reference frame 1:

$$P_1$$

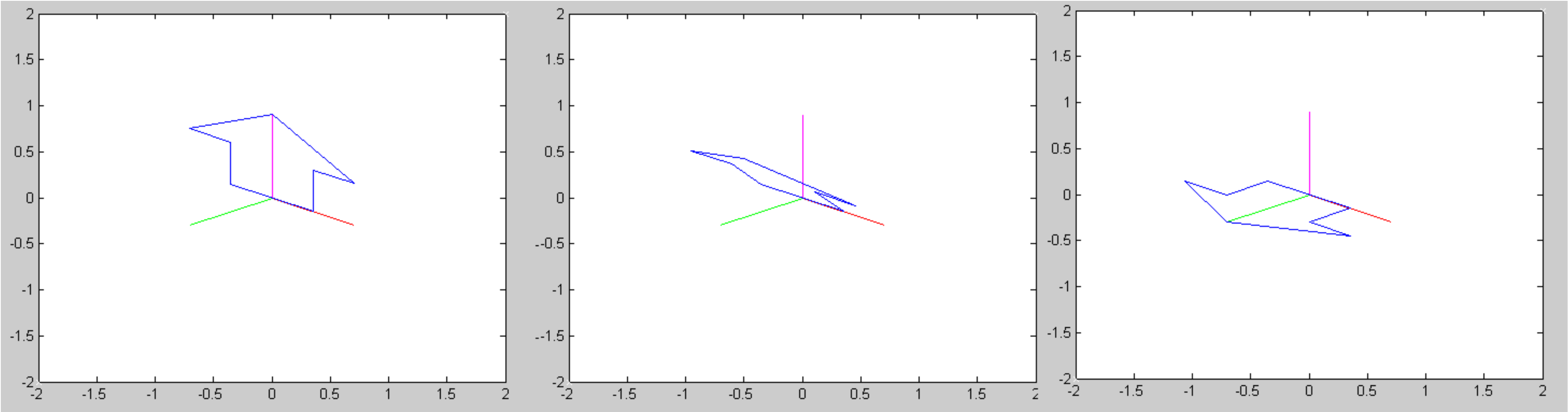
Relative to the zero reference frame, the points are:

$$P_0 = T_{01} P_1$$

$$P_0 = (T_{10})^{-1} P_1$$

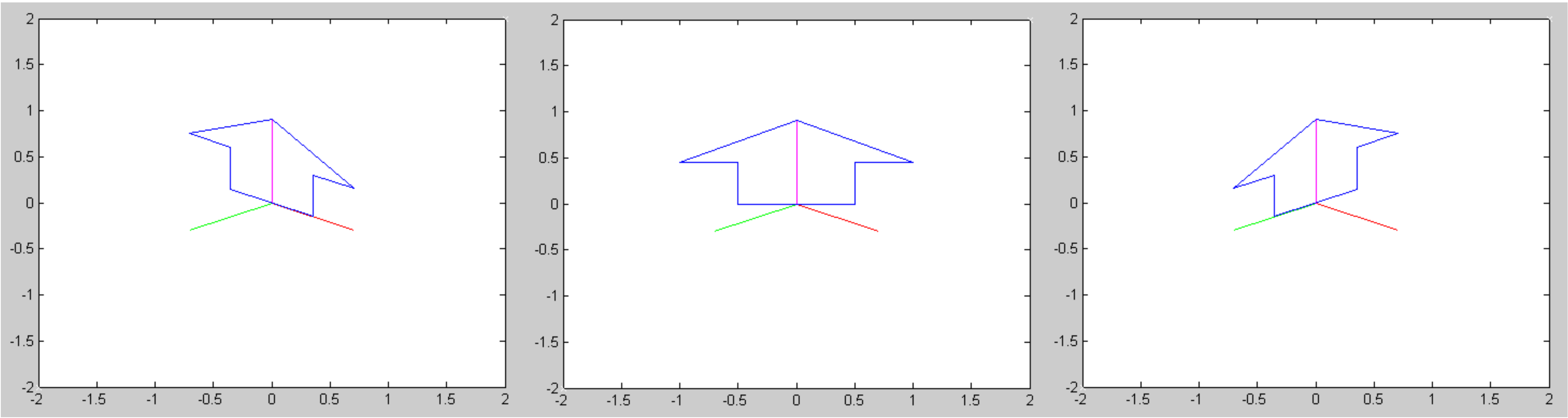
Rotate the Object About the Y Axis

```
c = cos(5*pi/180);
s = sin(5*pi/180);
Ry = [c,0,-s,0;0,1,0,0;s,0,c,0;0,0,0,1];
T01 = inv(Ry);

T = eye(4,4);
for i=1:1000
   T = T01*T;
   Display3D(T*ARROW, Tdisp);
   pause(0.01);
   end
```



Rotate the Object About the Z Axis

```
c = cos(5*pi/180);
s = sin(5*pi/180);
Rz = [c,s,0,0;-s,c,0,0;0,0,1,0;0,0,0,1];
T01 = inv(Rz);

T = eye(4,4);
for i=1:1000
   T = T01*T;
   Display3D(T*ARROW, Tdisp);
   pause(0.01);
   end
```



Homework #2:

- Define a shape other than an arror
- Rotate this object about
  - the x-axis
  - the y-axis
  - the z-axis

Demonstrate your code

- YouTube video
- Screen capture
- Share your screen on Zoom

Goal: Convince yourself that rotation matrices really do work.