LaGrangian Formulation of System Dynamics

Lecture #9

ECE 761: Robotics

Class taught at North Dakota State University
Department of Electrical and Computer Engineering

Please visit www.BisonAcademy.com for corresponding lecture notes, homework sets, and solutions.

LaGrangian Formulation of System Dynamics

- Robots are inherently nonlinear (lots of sine and cosine terms)
- To find the dynamics, we need a tool that can deal with nonlinear terms.
- LaGrangian dynamics is one such tool.

Idea:

- Identify the states of the system
- Specify the energy in the system
- Specify how the energy changes

If you know how energy moves through a system, you know its dynamics

Definitions:

KE Kinetic Energy in the system

PE Potential Energy

$\frac{\partial}{\partial t}$ The partial derivative with respect to 't'

$\frac{d}{dt}$ The full derivative with respect to t.

L Lagrangian = KE - PE

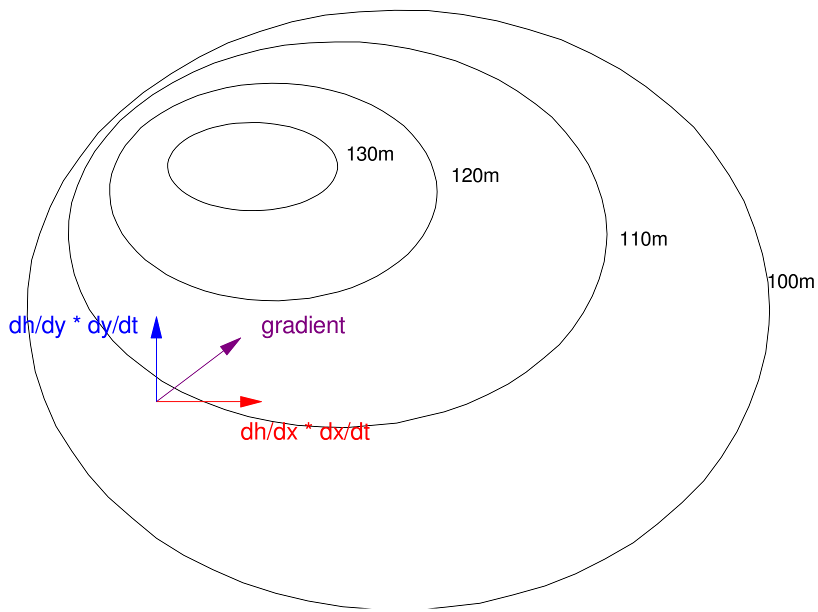
Full vs. Partial Derivatives

A full derivative contains partial derivatives

$$\frac{d}{dt} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t} + \ldots$$

Think of it as find how fast you are climbing

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial t}$$



Full vs. Partial Derivatives

When taking partial derivatives, treat all other terms as constants

- Otherwise you double count them

Example

$$A = xy^2 t$$

$$B = 3x\cos(t)$$

Find

$$C = \frac{d}{dt}\left(AB^2 t^3\right)$$

Solution: Uses partial and full derivatives

$$C = \left(\frac{\partial\left(AB^2 t^3\right)}{\partial A}\ \frac{\partial A}{\partial t}\right) + \left(\frac{\partial\left(AB^2 t^3\right)}{\partial B}\ \frac{\partial B}{\partial t}\right) + \left(\frac{\partial\left(AB^2 t^3\right)}{\partial t}\ \frac{\partial t}{\partial t}\right)$$

$$C = \left(B^2 t^3\right)\left(xy^2\right) + \left(2ABt^3\right)(-3x\sin(t)) + \left(3AB^2 t^2\right)(1)$$

Procedure for LaGrangian Dynamics:

1) Define the kinetic and potential energy in the system.
2) Form the Lagrangian:

$$L = KE - PE$$

3) The input is then

$$F_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}$$

where F_i is the input to state x_i. Note that

- If x_i is a position, F_i is a force.
- If x_i is an angle, F_i is a torque

Example: Rocket Dynamics

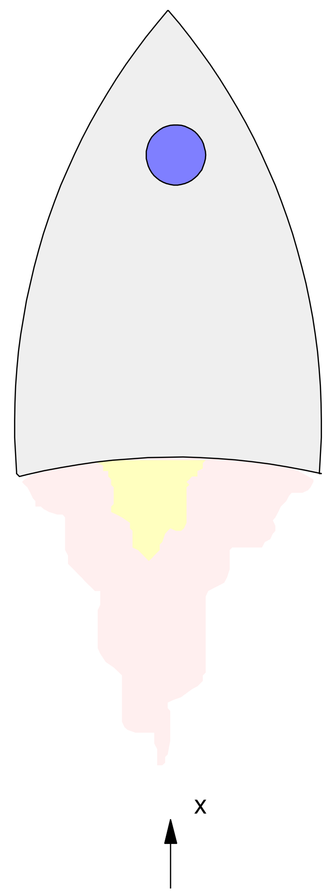
Step 1: Determine the potential and kinetic energy of the rocket

Potential Energy

$$PE = mgx$$

Kinetic Energy:

$$KE = \frac{1}{2}m\dot{x}^2$$



Step 2: Set up the LaGrangian

$$L = KE - PE$$

$$L = \frac{1}{2}m\dot{x}^2 - mgx$$

Step 3: Take the partials

$$L = \frac{1}{2}m\dot{x}^2 - mgx$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \frac{d}{dt}(m\dot{x}) - (-mg)$$

Take the full derivative with respect to t

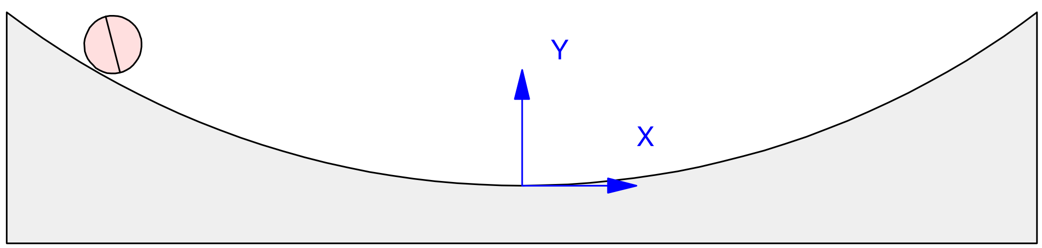
$$\boxed{F = m\ddot{x} + \dot{m}\dot{x} + mg}$$

Note that if the rocket is loosing mass you get the term $\dot{m}\dot{x}$. If you leave this term out, the rocket misses the target.

Example 2: Ball in a parabolic bowl

Find the dynamics when

$$y = \frac{1}{2}x^2$$



Step 1: Define the KW & PE

$$PE = mgy = \frac{1}{2}mgx^2$$

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}J\dot{\theta}^2$$

$$x = r\theta$$

which becomes

$$KE = \frac{1}{2}\left(m + \frac{J}{r^2}\right)\left(\dot{x}^2 + (x\dot{x})^2\right)$$

The inertia depends upon what type of ball you are using:

- $J = 0$ point mass with all the mass in the center
- $J = \frac{2}{5}mr^2$ solid sphere
- $J = \frac{2}{3}mr^2$ hollow sphere
- $J = mr^2$ hollow cylinder

Assume the ball is a solid sphere

$$KE = \frac{1}{2}\left(m + \frac{\frac{2}{5}mr^2}{r^2}\right)\left(\dot{x}^2 + (x\dot{x})^2\right)$$

$$KE = 0.7m\left(1^2 + x^2\right)\dot{x}^2$$

Step 2: Form the LaGrangian

$$L = KE - PE$$

$$L = 0.7m\left(1^2 + x^2\right)\dot{x}^2 - \tfrac{1}{2}mgx^2$$

Step 3: Take the partials.

$$F = \tfrac{d}{dt}\left(\tfrac{\partial L}{\partial \dot{x}}\right) - \left(\tfrac{\partial L}{\partial x}\right)$$

$$F = \tfrac{d}{dt}\left(1.4m\left(1 + x^2\right)\dot{x}\right) - \left(1.4mx\dot{x}^2 - mgx\right)$$

Take the full derivative

$$F = \frac{d}{dt}\left(1.4m\left(1+x^2\right)\dot{x}\right) - \left(1.4mx\dot{x}^2 - mgx\right)$$

$$F = \frac{\partial}{\partial x}\left(1.4m\left(1+x^2\right)\dot{x}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial \dot{x}}\left(1.4m\left(1+x^2\right)\dot{x}\right)\frac{\partial \dot{x}}{\partial t} - \left(1.4mx\dot{x}^2 - mgx\right)$$

$$F = (2.8mx\dot{x})\dot{x} + \left(1.4m\left(1+x^2\right)\right)\ddot{x} - \left(1.4mx\dot{x}^2 - mgx\right)$$

Simplifying

$$F = 1.4mx\dot{x}^2 + 1.4m\left(1+x^2\right)\ddot{x} + mgx$$

If F = 0

$$\ddot{x} = -\left(\frac{\left(1.4\dot{x}^2+g\right)x}{1.4\left(1^2+x^2\right)}\right)$$

Matlab Code (Ball.m)

```
while(t < 100)

   ddx = -( 1.4*dx*dx + 9.8) * x / ( 1.4*(1 + x*x) );

   dx = dx + ddx*dt;
   x = x + dx*dt;

    y = 0.5*x*x;

    x1 = [-2:0.01:2]';
    y1 = 0.5* (x1 .^ 2);

 % draw the ball
    i = [0:0.01:1]' * 2 * pi;
    xb = 0.05*cos(i) + x;
    yb = 0.05*sin(i) + 0.5*x^2 + 0.05 + 0.02*abs(x);

 % line through the ball
    q = [0, pi] - x/0.05;
    xb1 = 0.05*cos(q) + x;
    yb1 = 0.05*sin(q) + 0.5*x^2 + 0.05 + 0.02*abs(x);

    plot(x1,y1,'b', xb, yb, 'r', xb1, yb1, 'r');

    pause(0.01);
    end
```

Homework #9:

Determine the dynamics of a ball rolling in a different bowl

- Determine the kinetic energy
- Determine the potential energy
- Form the LaGrangian
- Take full and partial derivatives to determine the dynamics

Optional:

- Modify the bowl program to simulate the ball rolling freely